COL866: Special Topics in Algorithms	Fall 2023
Assignment 3	
Total points: 100	Deadline: Oct 26 (Thursday)

1. [**20 points**] An instance with indivisible chores is said to have *binary* valuations if for every agent i and for every chore j, it holds that $v_{i,j} \in \{0, -1\}$. Prove that determining whether a given chores instance with binary and additive valuations admits an envy-free allocation is NP-complete.

   Hint: You may reduce from the NP-complete problem `SET SPLITTING`, which is defined as follows: Given a finite set S and a family $\mathcal{F}$ of subsets of S, does there exist a partition of the set S into S_1 and S_2 such that every element of $\mathcal{F}$ is split by this partition (i.e., no element of $\mathcal{F}$ is completely contained in S_1 or in S_2)?

   Membership in NP is clear since checking whether a given allocation is envy-free can be done in polynomial time.

   To show NP-hardness, we will show a reduction from `SET SPLITTING`. Given an instance $\langle S, \mathcal{F} \rangle$ of `SET SPLITTING`, we will construct an instance of the fair division problem as follows: Let $s := |S|$ and $r := |\mathcal{F}|$. Note that we can assume, without loss of generality, that $r \geqslant s$ (otherwise, one can add copies of the set S to the family $\mathcal{F}$ without changing the solution). It will be helpful to think of the elements of the set S as *vertices*, the members of the family $\mathcal{F}$ as *hyperedges*, and membership in one of the sets S_1 or S_2 in the solution of `SET SPLITTING` as being assigned the *color* 1 or 2.

   The fair division instance consists of $r + 2$ agents, which includes r *edge* agents $e_1, \ldots, e_r$ and two *color* agents c_1 and c_2. Additionally, there are $r + s$ chores, which includes r *dummy* chores $D_1, \ldots, D_r$ and s *vertex* chores $V_1, \ldots, V_s$.

   The valuations are defined as follows: Each dummy chore is valued at -1 by all (edge and color) agents. Each vertex chore V_i is valued at -1 by those edge agents e_j whose corresponding hyperedge is adjacent to the vertex v_i in the `SET SPLITTING` instance, and is valued at 0 by all other edge agents. The color agents value each vertex chore at 0.

   We will now argue the equivalence of the solutions.

   `SET SPLITTING` $\Rightarrow$ `ENVY-FREE CHORE ALLOCATION` :
   Suppose there exists a partition of the set S into S_1 and S_2 such that each member of the family $\mathcal{F}$ is split by this partition. (Equivalently, suppose there exists a 2-coloring of associated hypergraph such that each hyperedge sees both colors.) Then, an envy-free allocation can be constructed as follows: Each edge agent is assigned exactly one

dummy chore. Additionally, if the vertex v_i is assigned the color $\ell \in \{1, 2\}$ in the `SET SPLITTING` solution, then the corresponding chore V_i is assigned to the color agent c_ℓ.

The above allocation assigns each chore to exactly one agent and is therefore feasible. Furthermore, it is also envy-free for the following reason: Each color agent only receives vertex chores which it values at 0; thus, it does not envy any other agent. An edge agent does not envy another edge agent because the dummy chores are valued identically at -1. Furthermore, an edge agent also does not envy any of the color agents because by the splitting property, each color agent gets at least one vertex chore that is valued at -1 by the edge agent.

`ENVY-FREE CHORE ALLOCATION` $\Rightarrow$ `SET SPLITTING` :
Now suppose there exists an envy-free allocation, say A. Then, it can argued that neither of the color agents are assigned a dummy chore. This is because assigning a dummy chore to a color agent will give it a utility of -1, and in order to compensate for the envy, *every* other agent will need to be assigned a chore that the color agent values at -1. This, however, is impossible as there are $r + 1$ other agents but only $r - 1$ such chores. Thus, all dummy chores must be allocated among the edge agents in the envy-free allocation A.

We will now argue that no edge agent receives more than one dummy chore under A. Suppose, for contradiction, that an edge agent e_i receives two or more dummy chores. Then, due to envy-freeness, every other agent must receive at least two chores that e_i values at -1. There are $r + 1$ other agents in total, requiring there to be at least $2r + 2$ such chores. However, the total number of chores is $r + s \leqslant 2r$ (recall that $s \leqslant r$), which is insufficient. Thus, each edge agent receives at most one dummy chore. Furthermore, since the number of dummy chores and edge agents is equal, we get that each edge agent receives exactly one dummy chore.

Since each edge agent receives exactly one dummy chore, in order to compensate for the envy, each color agent must be assigned at least one vertex chore that the edge agent values at -1. The desired 2-coloring of the hypergraph (equivalently, desired partition of the set S) can now be inferred from the allocation A; in particular, the vertex chores that are assigned to an edge agent can be arbitrarily put into S_1 or S_2.

This problem was based on (Bhaskar et al., 2021, Section A.1, Theorem 2).

2. [**15 points**] Consider a weighted generalization of the fair division problem with indivisible goods, where each agent i is associated with a weight $w_i > 0$ such that $\sum_i w_i = 1$.

   Let us define weighted generalizations of the fairness notions seen in the lectures as follows: A randomized allocation X is said to be *weighted envy-free* (wEF) if, for any pair of agents i and j, we have $\frac{\mathbb{E}[v_i(X_i)]}{w_i} \geqslant \frac{\mathbb{E}[v_i(X_j)]}{w_j}$. Similarly, a deterministic allocation A is said to satisfy *weighted envy-freeness up to one good* (wEF1) if, for any pair of agents i and j with $A_j \neq \emptyset$, there exists a good $g \in A_j$ such that $\frac{v_i(A_i)}{w_i} \geqslant \frac{v_i(A_j \setminus \{g\})}{w_j}$.

Prove or disprove: For two agents with additive valuations over indivisible goods, there always exists a randomized allocation that is ex-ante wEF and ex-post wEF1.

We will disprove the statement by means of the following counterexample: Consider an instance with two agents a_1 and a_2 with weights $w_1 = 0.5 + \varepsilon$ and $w_2 = 0.5 - \varepsilon$ and two items g_1 and g_2. Agent a_1 has equal positive values for the two items, and agent a_2 also has equal positive values for the two items that may be different from a_1's values.

Observe that there are only two wEF1 allocations in this instance—namely, $A = (\{g_1\}, \{g_2\})$ and $B = (\{g_2\}, \{g_1\})$; in particular, assigning both items to a single agent violates wEF1 from the other agent's perspective.

Now, in order for a randomized allocation X to satisfy ex-post wEF1, it must be supported only over A and B. Under any such distribution, agent a_1's expected utility for its own randomized bundle X_1 is equal to its expected utility for agent a_2's randomized bundle X_2, which violates ex-ante wEF.

This problem was based on (Aziz et al., 2023, Theorem 4.1).

3. [**10 points**] For the voting instance given below, compute the election winners under Plurality, Borda, Plurality-with-runoff, STV, Copeland, and Schulze voting rules. Ties are broken according to the lexicographic ordering $a \succ b \succ c \succ d \succ e$.

$$\begin{array}{rl} 3 \text{ voters:} & c \succ d \succ b \succ a \succ e \\ 8 \text{ voters:} & c \succ e \succ b \succ d \succ a \\ 18 \text{ voters:} & d \succ e \succ c \succ b \succ a \\ 22 \text{ voters:} & e \succ c \succ b \succ d \succ a \\ 16 \text{ voters:} & b \succ d \succ c \succ e \succ a \\ 33 \text{ voters:} & a \succ b \succ c \succ d \succ e \end{array}$$

Plurality: a (Plurality scores are a 33, b 16, c:11, d:18, e:22)

Plurality-with-runoff: e (since a and e survive round 1 and e beats a in a head-to-head).

Borda count: b (Borda scores are a:135, b:247, c:244, d:192, e:182)

Copeland and Schulze: c (since c is a Condorcet winner and both Copeland and Schulze satisfy the Condorcet criterion)

STV: d

Acknowledgement: The voting instance used in this problem is based on the one in Lecture 1 of the course on "Optimized Democracy" by Ariel Procaccia.

4. [**15 points**] On the YouTube channel of the online COMSOC video seminar:

   `https://www.youtube.com/channel/UCa_l2EzXiJxzfZKtu2mTkdA/videos`

   pick any one talk of your choice and summarize it in no more than 500 words. (Each video on the channel contains two or more talks; you only need to pick one talk that is sufficiently different from your project topic.) Feel free to use mathematical notation, examples or pictures in your summary if needed. Also provide the bibliographical information of the article(s) that the talk is based on.

5. [**30 points**] The Gibbard-Satterthwaite theorem establishes that no "reasonable" voting rule is strategyproof, meaning there is always a worst-case profile where some voter could misreport and improve. However, this result does not inform us about *how often* such bad profiles arise. In this exercise, we will experimentally analyze the frequency of manipulability of certain voting rules.

   Say there are n voters and m candidates. Design an experiment where the preference of each voter is generated independently and uniformly at random from among the $m!$ possible rankings over the candidates. Consider the following three voting rules: Plurality, Borda, and Copeland. Compare these rules in terms of the *fraction* of manipulable preference profiles (i.e., preference profiles where at least one voter can improve by misreporting). Assume throughout that ties are broken lexicographically.

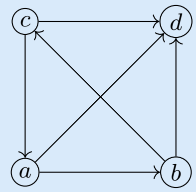
   Clearly explain the (a) *hypothesis* (e.g., which voting rule did you expect to be the "most frequently manipulable" before you started the experiments and why), (b) *experimental setup* (e.g., what values of n and m did you consider, how many preference profiles did you sample for each setting of n and m and what were the reasons for these choices, which algorithm did you use to determine the manipulability of a profile), (c) *experimental observations*, and (d) *inference*.

6. [**10 points**] A *tournament* graph $G = (V, E)$ is a complete directed graph where, for any pair of vertices $i, j \in V$, either there is a directed edge from i to j or there is one from j to i.

   Given an *arbitrary* tournament graph $G = (V, E)$ on $|V| = m$ vertices, construct an election with m candidates and at most m^2 voters such that, for any pair of candidates i and j, a majority of voters prefer i over j if and only if there is a directed edge from i to j in G.

   In other words, show that *any* preference pattern can be realized as the majority vote of some group of voters.

   > Here's a proof sketch for a graph with four vertices (the same idea can be extended to any given graph):



For the edge $a \to b$, create the following two votes:

$$v_1 : \boldsymbol{a} \succ \boldsymbol{b} \succ c \succ d$$

$$v_2 : d \succ c \succ \boldsymbol{a} \succ \boldsymbol{b}$$

This has the effect of *reinforcing* the comparisons between a and b and *canceling out* the comparisons between all other pairs of candidates. Similarly,

$$v_3 : \boldsymbol{c} \succ \boldsymbol{a} \succ b \succ d$$

$$v_4 : d \succ b \succ \boldsymbol{c} \succ \boldsymbol{a}$$
$$v_5 : \boldsymbol{a} \succ \boldsymbol{d} \succ b \succ c$$
$$v_6 : c \succ b \succ \boldsymbol{a} \succ \boldsymbol{d}$$
$$v_7 : \boldsymbol{b} \succ \boldsymbol{c} \succ a \succ d$$
$$v_8 : d \succ a \succ \boldsymbol{b} \succ \boldsymbol{c}$$
$$v_9 : \boldsymbol{b} \succ \boldsymbol{d} \succ a \succ c$$
$$v_{10} : c \succ a \succ \boldsymbol{b} \succ \boldsymbol{d}$$
$$v_{11} : \boldsymbol{c} \succ \boldsymbol{d} \succ a \succ b$$
$$v_{12} : b \succ a \succ \boldsymbol{c} \succ \boldsymbol{d}$$

References

Haris Aziz, Aditya Ganguly, and Evi Micha. Best of Both Worlds Fairness under Entitlements. In *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems*, pages 941–948, 2023. (Cited on page 3)

Umang Bhaskar, AR Sricharan, and Rohit Vaish. On Approximate Envy-Freeness for Indivisible Chores and Mixed Resources. *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, 2021. (Cited on page 2)