COL866: Special Topics in Algorithms **Fall 2023**

Assignment 1

Total points: 100 Deadline: Aug 21 (Monday)

1. a) [**5 points**] Show that under the men-proposing DA algorithm, there is always at least one woman who receives exactly one proposal.

   b) [**5 points**] Suppose there are n men and n women. As a function of n, what is the maximum number of proposals that can be made during the DA algorithm? (Hint: Use the above result.)

   c) [**5 points**] For an arbitrary n, construct an instance where the number of proposals made by men under the DA algorithm matches the bound shown by you above.

2. [**10 points**] In this exercise, we will show that the strategy of repeatedly fixing blocking pairs may not give a stable matching. Consider the matching instance given below:

$$\begin{array}{llll} m_1: & w_2 & w_1 & w_3 \\ m_2: & w_1 & w_2 & w_3 \\ m_3: & w_1 & w_2 & w_3 \end{array} \qquad \begin{array}{llll} w_1: & m_1 & m_3 & m_2 \\ w_2: & m_3 & m_1 & m_2 \\ w_3: & m_3 & m_1 & m_2 \end{array}$$

   Show that there is a cyclic sequence of unstable matchings in the above instance such that each matching in sequence can be obtained from its predecessor by "fixing" a blocking pair (i.e., by matching the blocking agents with each other, and also matching their previously assigned partners with each other).

3. Define the *cost* of a stable matching as sum of ranks of matched partners of all agents (men and women). For example, for the stable matching given by the underlined outcomes in the instance below, the cost is 1+1+2+1+4+1+2+3 = 15.

$$\begin{array}{lllll} m_1: & \underline{w_3} & w_2 & w_1 & w_4 \\ m_2: & \underline{w_1} & w_4 & w_2 & w_3 \\ m_3: & w_2 & \underline{w_4} & w_1 & w_3 \\ m_4: & \underline{w_2} & w_1 & w_3 & w_4 \end{array} \qquad \begin{array}{lllll} w_1: & m_4 & m_3 & m_1 & \underline{m_2} \\ w_2: & \underline{m_4} & m_3 & m_2 & m_1 \\ w_3: & m_3 & \underline{m_1} & m_2 & m_4 \\ w_4: & m_2 & m_1 & \underline{m_3} & m_4 \end{array}$$

   a) [**5 points**] For a general n, construct an instance with n men and n women and a stable matching for that instance with cost $n(n+1)$.

   b) [**15 points**] Prove that for any given preferences of n men and n women, the cost of *any* stable matching is at most $n(n+1)$.

4. [**10 points**] Recall from Lecture 3 that when a woman manipulates *optimally* under the men-proposing DA algorithm, the resulting matching is guaranteed to be stable with respect to the true preferences.

Define *suboptimal* manipulation by a woman as a misreport where she gets a partner who, according to her true list, is better than her true match but worse than her optimally manipulated match (that is, a suboptimal manipulation is better than telling the truth, but not as good as optimal manipulation). Provide an instance where the DA matching after suboptimal manipulation is *not* stable with respect to the true preferences.

5. Consider the stable matching instance given below:

$$
\begin{array}{lllll}
m_1\colon & w_1 & w_2 & w_3 & w_4 \\
m_2\colon & w_2 & w_1 & w_4 & w_3 \\
m_3\colon & w_3 & w_4 & w_1 & w_2 \\
m_4\colon & w_4 & w_3 & w_2 & w_1
\end{array}
\qquad\qquad
\begin{array}{lllll}
w_1\colon & m_3 & m_4 & m_2 & m_1 \\
w_2\colon & m_4 & m_3 & m_1 & m_2 \\
w_3\colon & m_1 & m_2 & m_4 & m_3 \\
w_4\colon & m_2 & m_1 & m_3 & m_4
\end{array}
$$

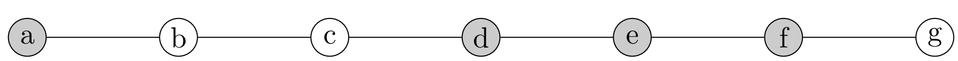
   a) [**10 points**] List all matchings that are stable for this instance.

   b) [**5 points**] Illustrate the lattice of stable matchings for the above instance. (You may find it convenient to use the "vector of ranks of matched partners" notation discussed in Lecture 2.)

   c) [**5 points**] Identify the men-optimal, women-optimal, median, egalitarian, and minimum regret matchings. If multiple matchings satisfy a given criterion (e.g., median, egalitarian, etc.), identify all matchings that do.

6. [**10 points**] Prove or disprove the following statement for many-to-one stable matchings: It is impossible to have one stable matching where a hospital is matched only with its 1st and 4th choices, and another stable matching in which it is matched only with its 2nd and 3rd choices.

7. Consider a two-hospital kidney exchange setting with seven patient-donor pairs. The compatibility graph among these patient-donor pairs is as shown below. Each node in this graph represents a patient-donor pair and each edge denotes the possibility of a two-way exchange between adjacent nodes (we will assume that only two-way exchanges are allowed). The shaded (respectively, unshaded) nodes belong to hospital 1 (respectively, hospital 2).



   Suppose the hospitals are strategic agents who are only interested in getting as many of their own nodes matched as possible. Each hospital can choose to hide a subset of its patient-donor pairs from the centralized exchange (and match them internally if possible). Thus, hospital 1 (respectively, hospital 2) can hide any subset of the shaded (respectively, unshaded) nodes. The centralized exchange only sees the nodes revealed by the two hospitals as well as the edges (if any) between pairs of revealed nodes.

   a) [**5 points**] Suppose the centralized exchange uses a deterministic maximum matching algorithm. Show that under any such algorithm, some hospital will have an incentive to not reveal all of its nodes. (In other words, show that any maximum matching algorithm fails to be strategyproof.)

b) [**10 points**] Now suppose the centralized exchange uses a deterministic strategyproof algorithm.[1] Show that any such algorithm, on some input, will be forced to match at most half as many nodes as the maximum matching algorithm. (In other words, no deterministic strategyproof algorithm can guarantee more than half the size of a maximum matching.)

[1] Showing the existence of a strategyproof algorithm is non-trivial. For the purpose of this exercise, you can simply assume that there exists some strategyproof algorithm.