






Lecture 9

Fairness and Efficiency




The Model

	(A)	(B)	(C)	(D)	(E)
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

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Additive
valuations

$$\begin{aligned} \triangle \{ \text{(B)} \text{(D)} \text{(E)} \} &= \triangle \{ \text{(B)} \} + \triangle \{ \text{(D)} \} + \triangle \{ \text{(E)} \} \\ &= 0 + 1 + 1 = 2 \end{aligned}$$



Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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My bundle is better
if (A) is removed



My bundle is better
if (C) is removed



(A)

(B)

(C)

4

1

2

1

1

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Allocation $A = (A_1, \dots, A_n)$ is EF1 if for every pair of agents i, k , there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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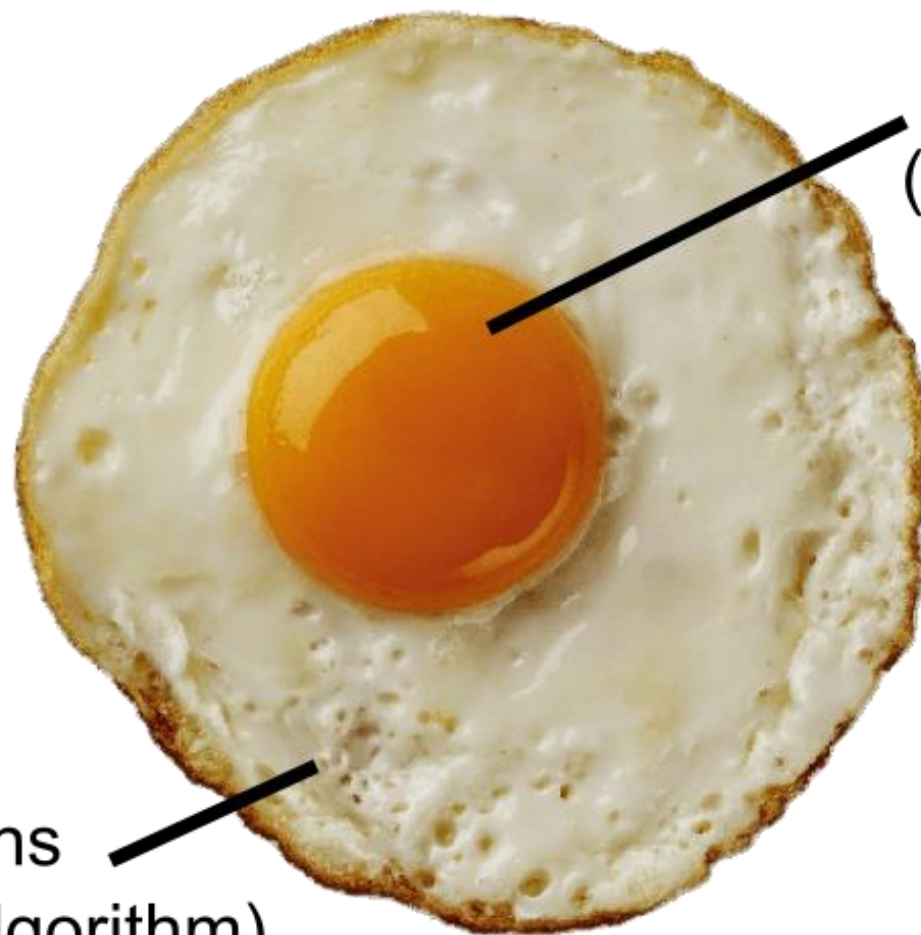
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Guaranteed to exist and efficiently computable

Last Time

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)



Monotone valuations
(Envy-cycle elimination algorithm)

A trivial way of achieving fairness: **Don't allocate anything!**

A bare minimum efficiency requirement: **Completeness**

"Obvious" Improvement

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	(A)	(B)	(C)	(D)	(E)
	4	3	1	1	1
	5	2	1	1	1

"Obvious" Improvement

	(A)	(B)	(C)	(D)	(E)
Red Triangle	4	3	1	1	1
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"obviously"
improved
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"Obvious" Improvement

	(A)	(B)	(C)	(D)	(E)
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"obviously"
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Strictly improving someone without hurting anyone else

WHEN A COMPLETE ALLOCATION



SIMPLY ISN'T ENOUGH



Pareto Optimality

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To make someone better off, someone else must be made worse off.



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Allocation **A** is *Pareto optimal* (PO) if no other allocation **B** Pareto improves it.

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Is EF1 compatible with Pareto optimality?

Round Robin Fails Pareto Optimality



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	4	3	1	1	1
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Pareto improved by

Envy-Cycle Elimination Fails Pareto Optimality

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Another natural strategy for EF1+PO

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Start with an EF1 allocation, and repeatedly make Pareto improvements to it.

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Exercise: Pareto improvement can fail to preserve EF1.



Nash Social Welfare

[Nash, 1950; Kaneko and Nakamura, 1979]

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A **Nash optimal** allocation is one that maximizes Nash social welfare.*

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*If optimal is 0, then find *any* largest set of agents who can simultaneously be given positive utility and maximize the geometric mean with respect to only those agents.

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *TEAC* 2019]

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Why PO?

Pareto improvement strictly improves NSW.

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *TEAC* 2019]

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Let $g^* \in \arg \min_{g \in A_k: v_i(\{g\}) > 0} v_k(\{g\})/v_i(\{g\})$.

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Does such a good g^* always exist?

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *TEAC* 2019]

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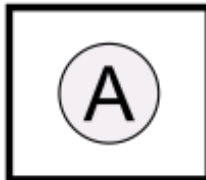




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We will show that transferring g^* from agent k to agent i improves NSW.

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$$\frac{NSW(B)}{NSW(A)} > 1$$

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$$\Leftrightarrow \left(1 - \frac{v_k(g^*)}{v_k(A_k)}\right) \cdot \left(1 + \frac{v_i(g^*)}{v_i(A_i)}\right) > 1$$

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By our choice of g^* :

$$\frac{v_k(g^*)}{v_i(g^*)} \leq \frac{\sum_{g \in A_k} v_k(g)}{\sum_{g \in A_k} v_i(g)} = \frac{v_k(A_k)}{v_i(A_k)}$$

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By our choice of g^* :

$$\frac{v_k(g^*)}{v_i(g^*)} \leq \frac{\sum_{g \in A_k} v_k(g)}{\sum_{g \in A_k} v_i(g)} = \frac{v_k(A_k)}{v_i(A_k)} \leftarrow \text{combining these}$$

By EF1 violation:

$$v_i(A_i) < v_i(A_k) - v_i(g^*)$$

Any Nash optimal allocation satisfies Pareto optimality (PO) and envy-freeness up to one good (EF1).

Why EF1?

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By EF1 violation:

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Ok, so an EF1+PO allocation always exists.

But what about computation?

Can a Nash optimal allocation be efficiently computed?

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[Lee, *Info. Proc. Lett.* 2017]

Maximizing Nash social welfare is APX-hard.

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- Even for bounded valuations

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Can an EF1+PO allocation be efficiently computed?

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An EF1+PO allocation can be computed in pseudopolynomial time.

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- Running time depends on $v_{i,j}$'s rather than $\log v_{i,j}$'s
- Polynomial time for bounded valuations
- A 0.69-approximation to Nash social welfare objective



spliddit



Share Rent

Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.

START >



Split Fare

Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.

START >



Assign Credit

Determine the contribution of each individual to a school project, academic paper, or business endeavor.

START >



Divide Goods



Distribute Tasks



Suggest an App

Next Time

Indivisible Chores



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