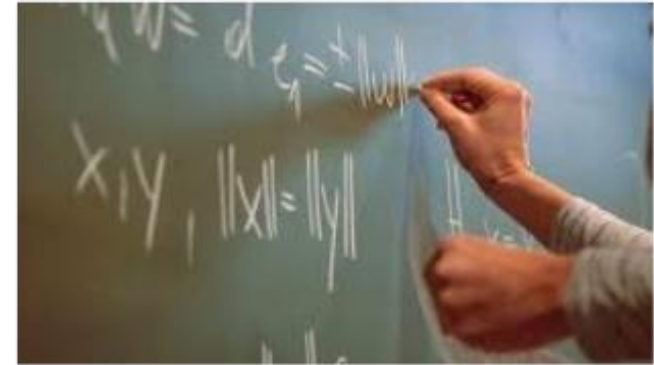


Lecture 8

Fair Allocation of Indivisible Goods



The Model

The Model



The Model

A

B




C

D




E






The Model

	(A)	(B)	(C)	(D)	(E)
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

The Model

	(A)	(B)	(C)	(D)	(E)
	4	1	2	2	2
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The Model

	(A)	(B)	(C)	(D)	(E)
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Additive
valuations

$$\begin{aligned} \triangle \{ \text{(B)} \text{(D)} \text{(E)} \} &= \triangle \{ \text{(B)} \} + \triangle \{ \text{(D)} \} + \triangle \{ \text{(E)} \} \\ &= 0 + 1 + 1 = 2 \end{aligned}$$

Envy-Freeness



[Gamow and Stern, 1958; Foley, 1967]

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Each agent prefers its own bundle over that of any other agent.



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	(A)	(B)	(C)
	4	1	2
	1	1	5

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	(A)	(B)	(C)
	4	1	2
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Each agent prefers its own bundle over that of any other agent.

	(A)	(B)	(C)
My bundle is the best	4	1	2
	1	1	5

Envy-Freeness [Gamow and Stern, 1958; Foley, 1967]

Each agent prefers its own bundle over that of any other agent.

	(A)	(B)	(C)
	4	1	2
 My bundle is the best	1	1	5

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My bundle is the best	4	1	2
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Allocation $A = (A_1, A_2, \dots, A_n)$ is EF if for every pair of agents i, k , we have $v_i(A_i) \geq v_i(A_k)$.

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Not guaranteed to exist (two agents, one good)

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Not guaranteed to exist (two agents, one good)



Checking whether an EF allocation exists is NP-complete



Envy-Freeness Up To One Good [Budish, 2011]

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Envy can be eliminated by removing some good in the envied bundle.



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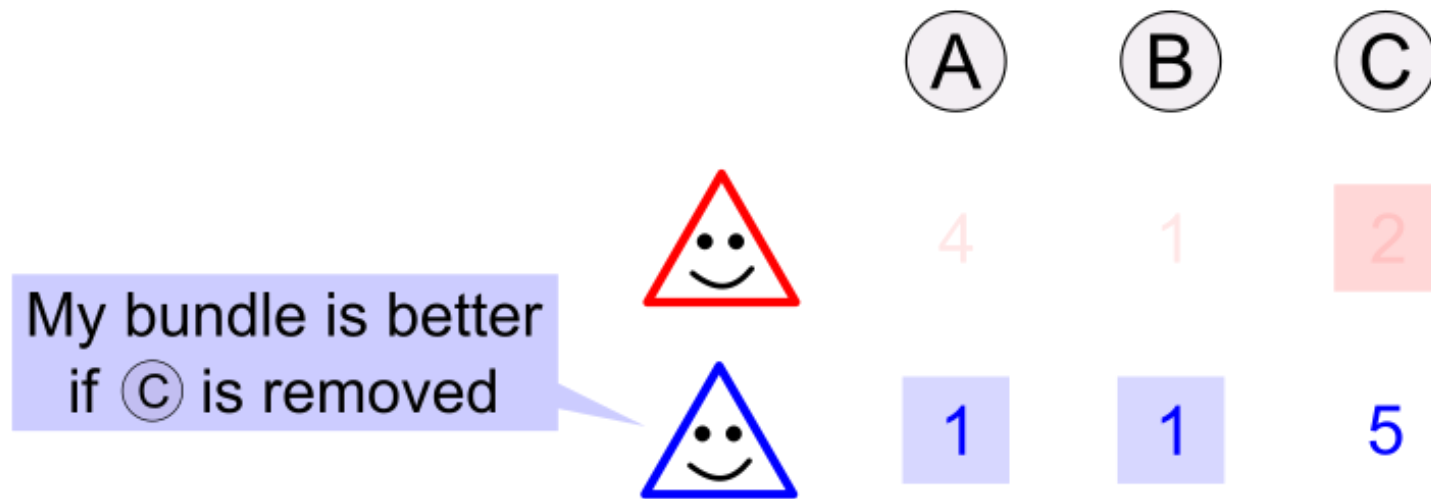
Envy can be eliminated by removing some good in the envied bundle.

My bundle is better
if (A) is removed

	(A)	(B)	(C)
Red Triangle	4	1	2
Blue Triangle	1	1	5

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My bundle is better
if (A) is removed



My bundle is better
if (C) is removed



(A)

(B)

(C)

4

1

2

1

1

5

(A)	(B)	(C)
4	1	2
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Allocation $A = (A_1, \dots, A_n)$ is EF1 if for every pair of agents i, k , there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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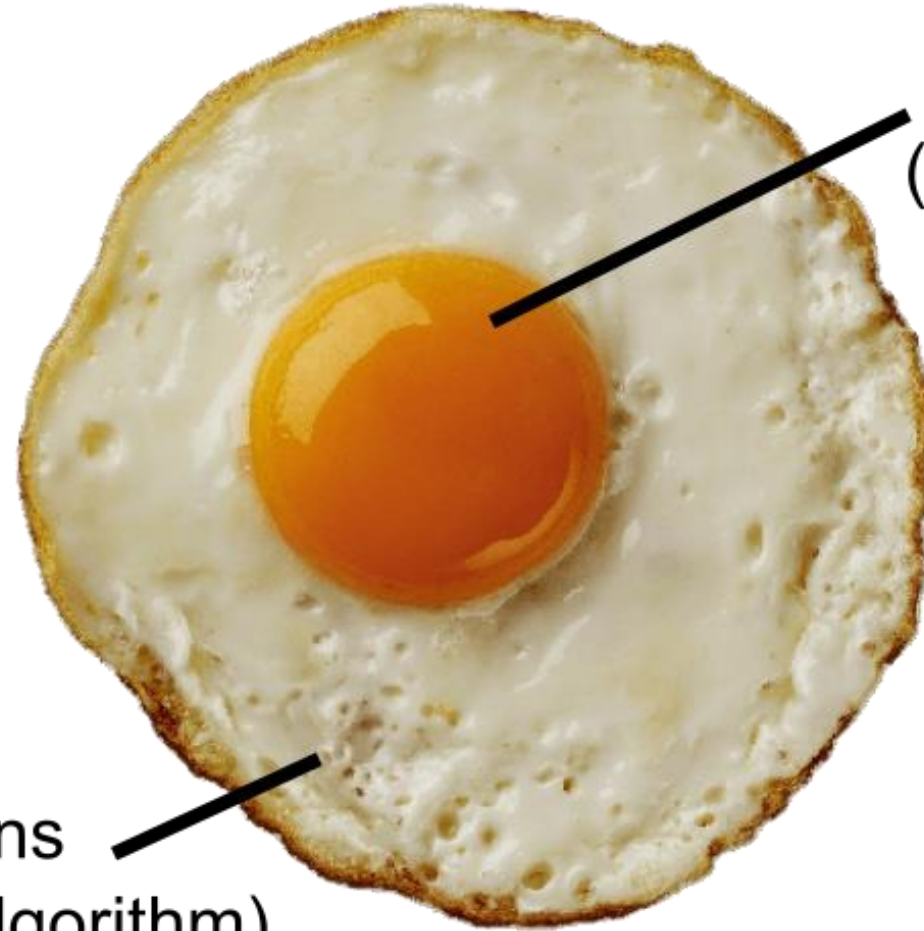
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Guaranteed to exist and efficiently computable

Coming Up

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

Round-robin algorithm

Round-robin algorithm



- Fix an ordering of the agents, say $a_1, a_2, a_3, \dots, a_n$.

Round-robin algorithm

- Fix an ordering of the agents, say $a_1, a_2, a_3, \dots, a_n$.
- Agents take turns according to the ordering $(a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n, \dots)$ to pick their favorite item from the set of remaining items.




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For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

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a_1 a_2 a_3 \dots a_n


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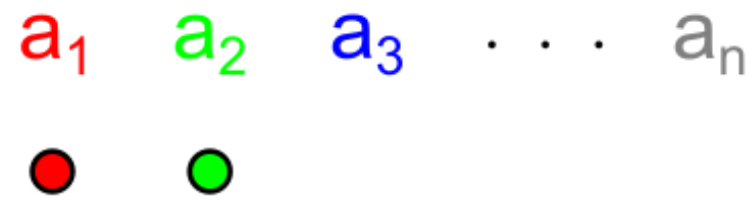
First round

For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

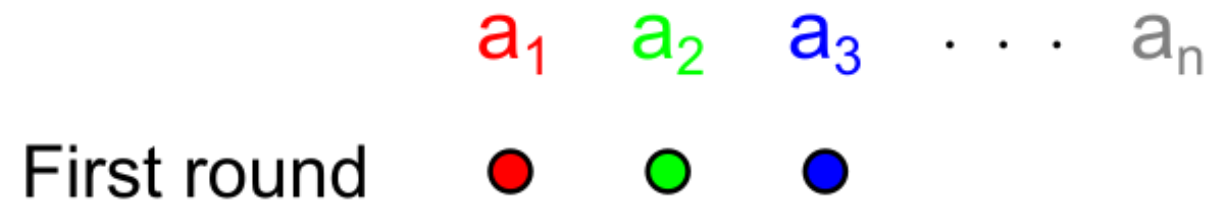
First round a_1 a_2 a_3 \dots a_n



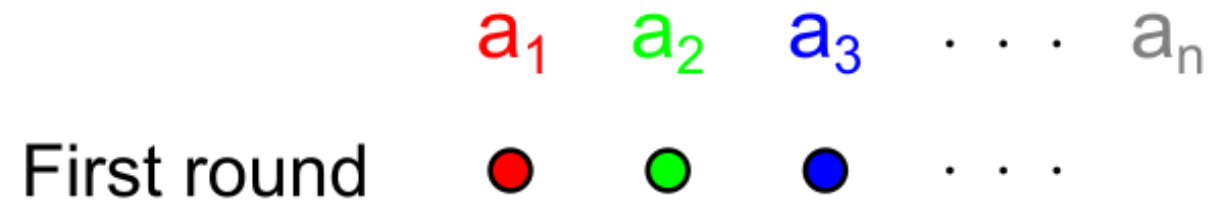
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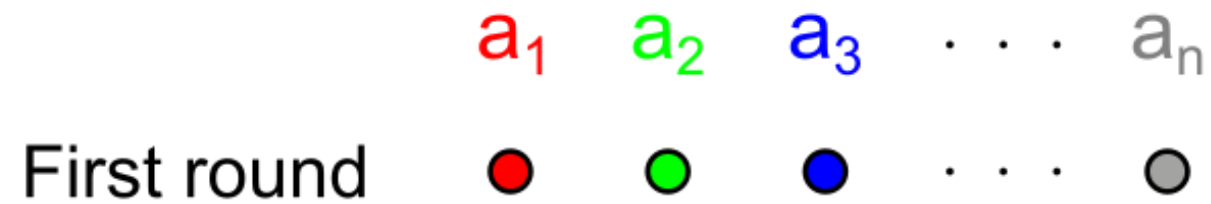
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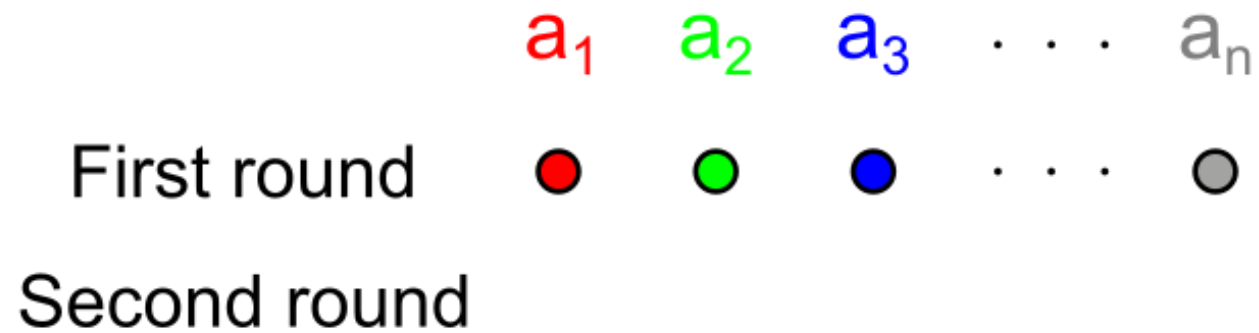
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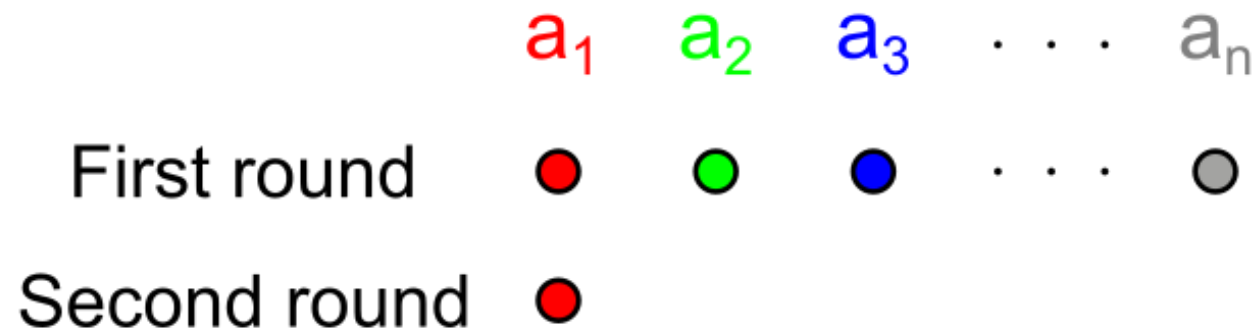
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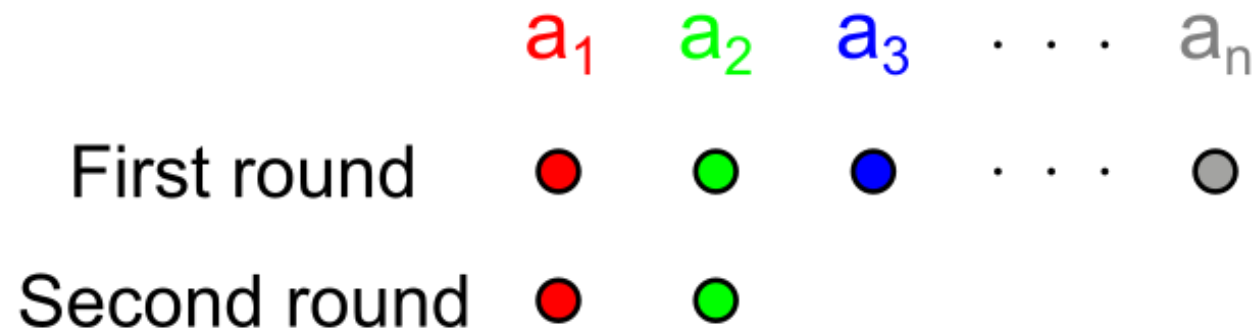
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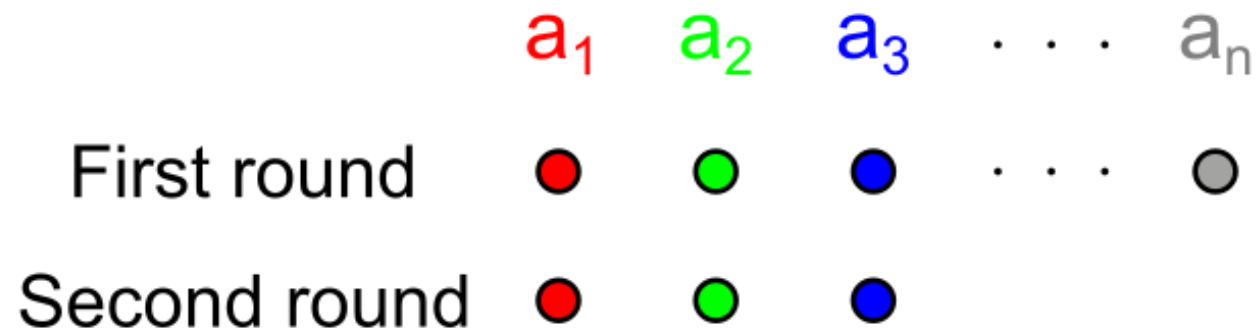
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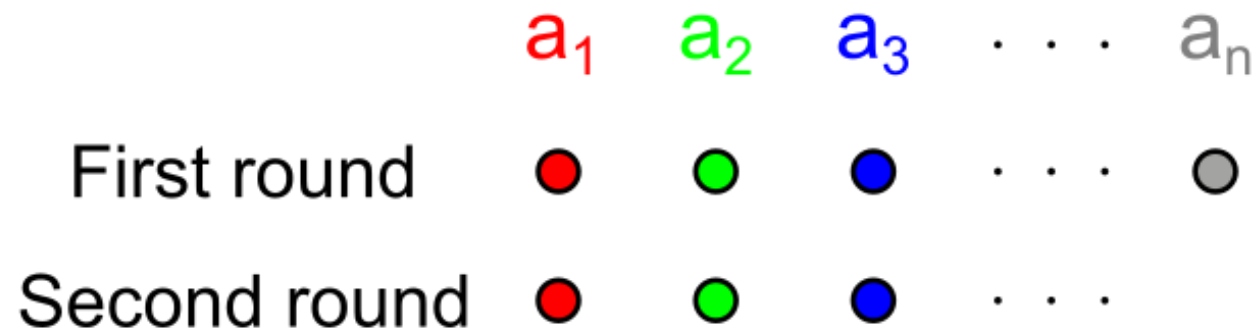
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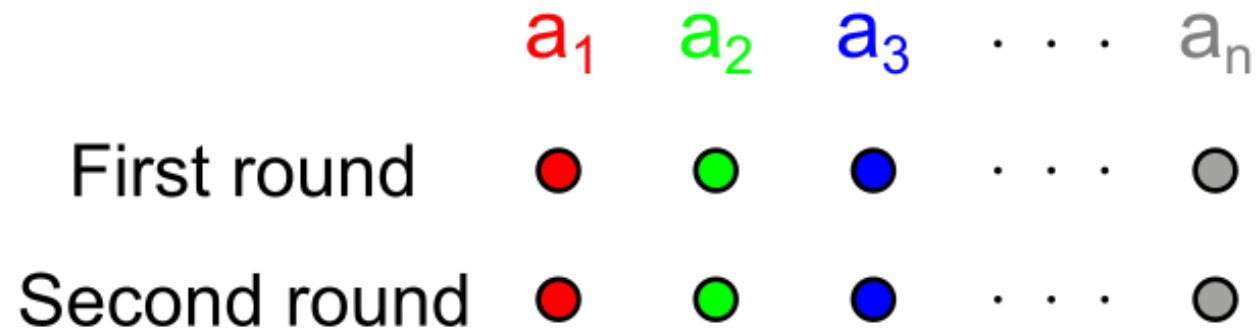
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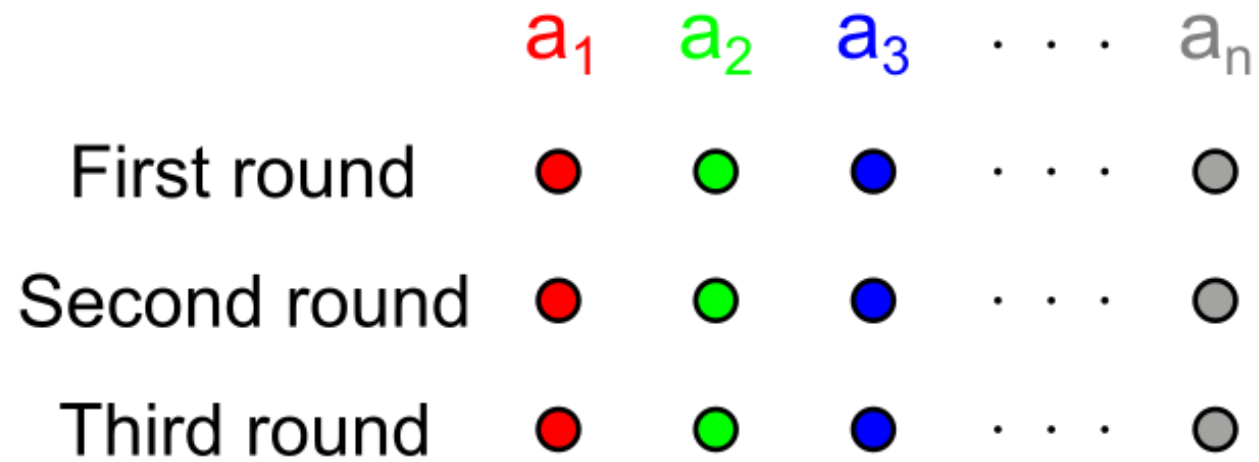
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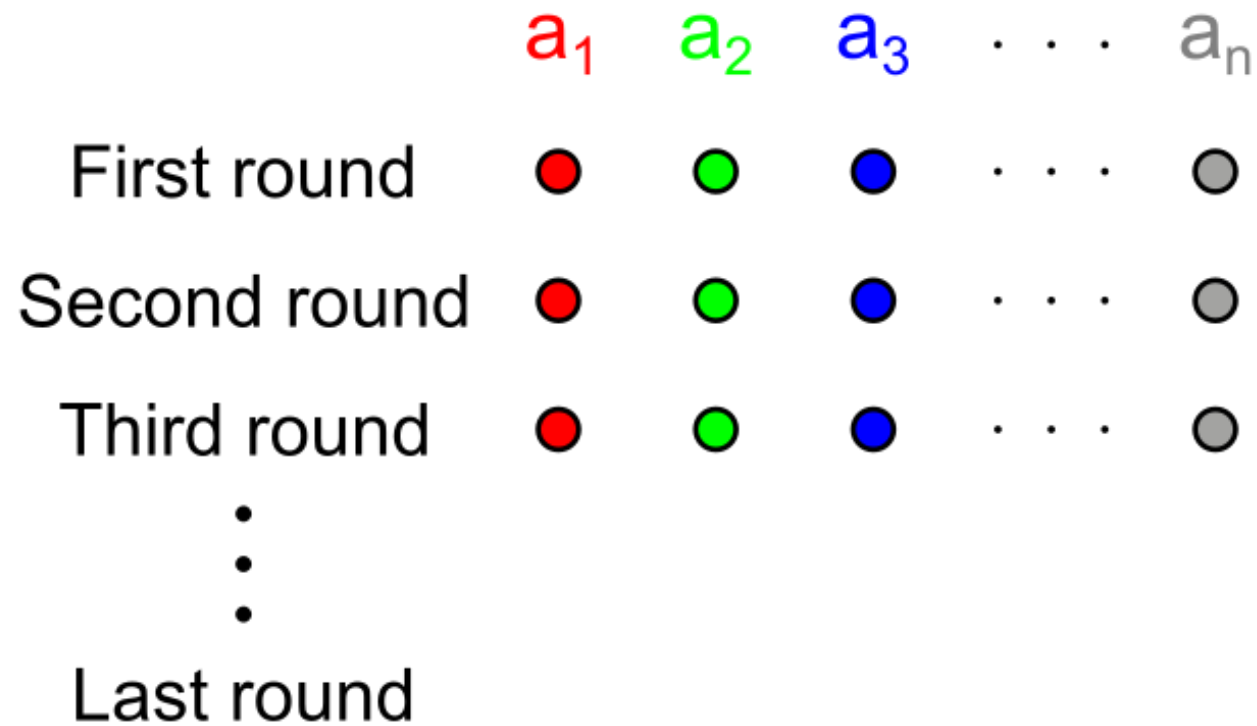
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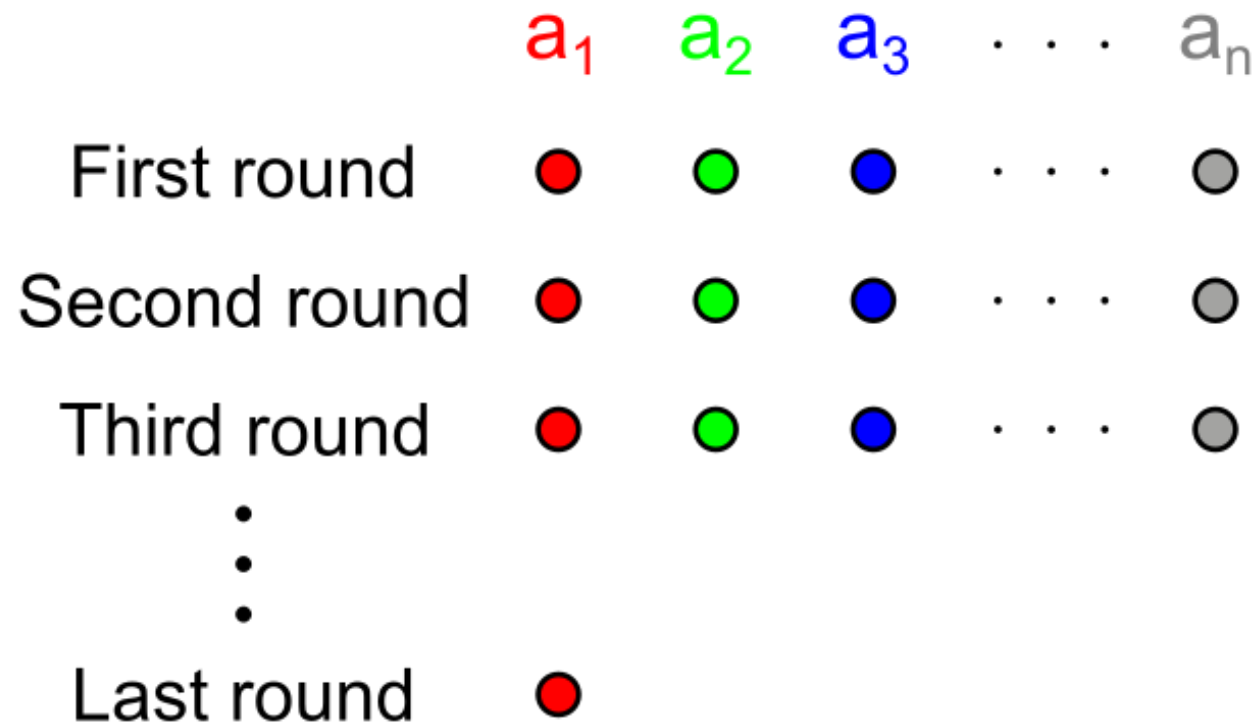
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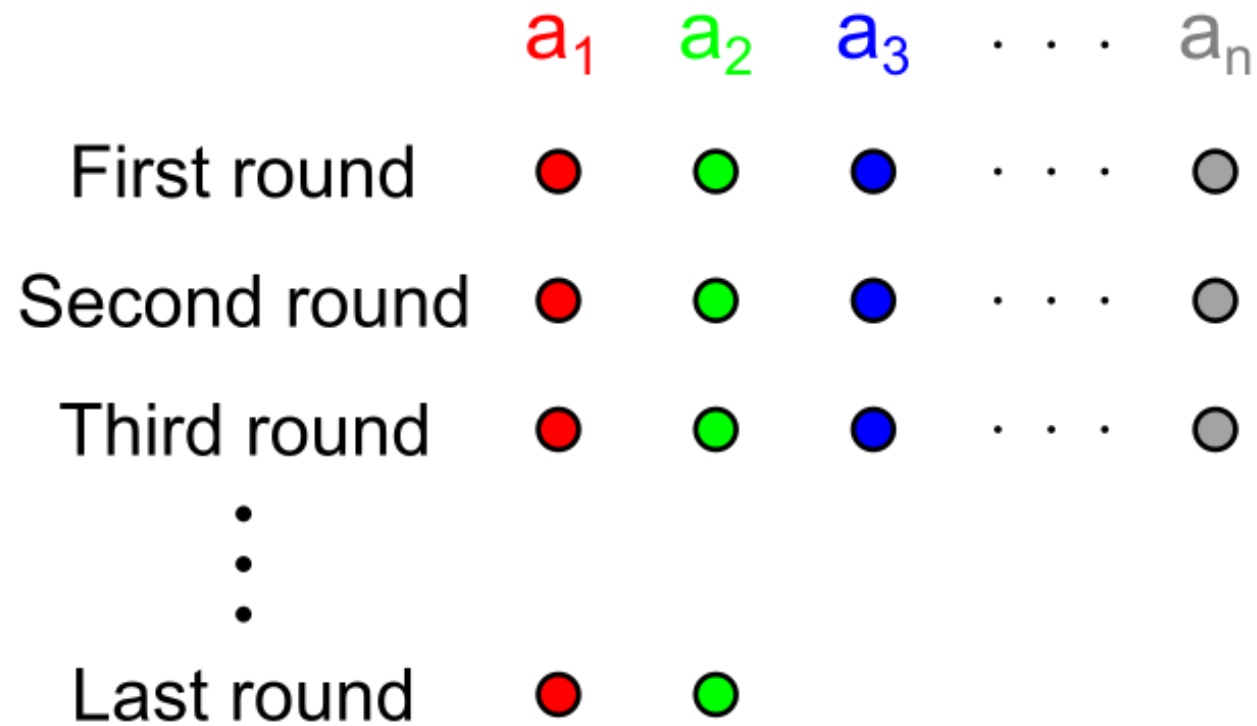
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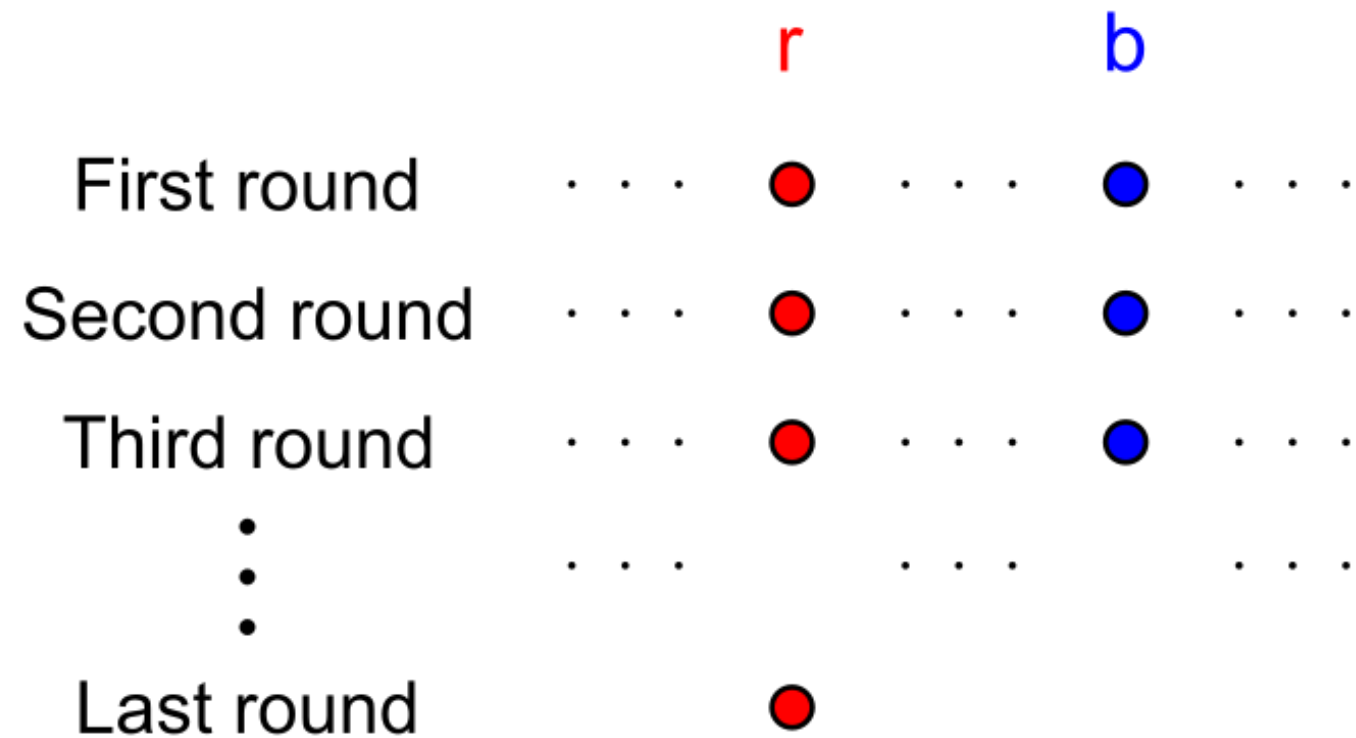


For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

Fix a pair of agents (r, b). Analyze envy of r towards b .

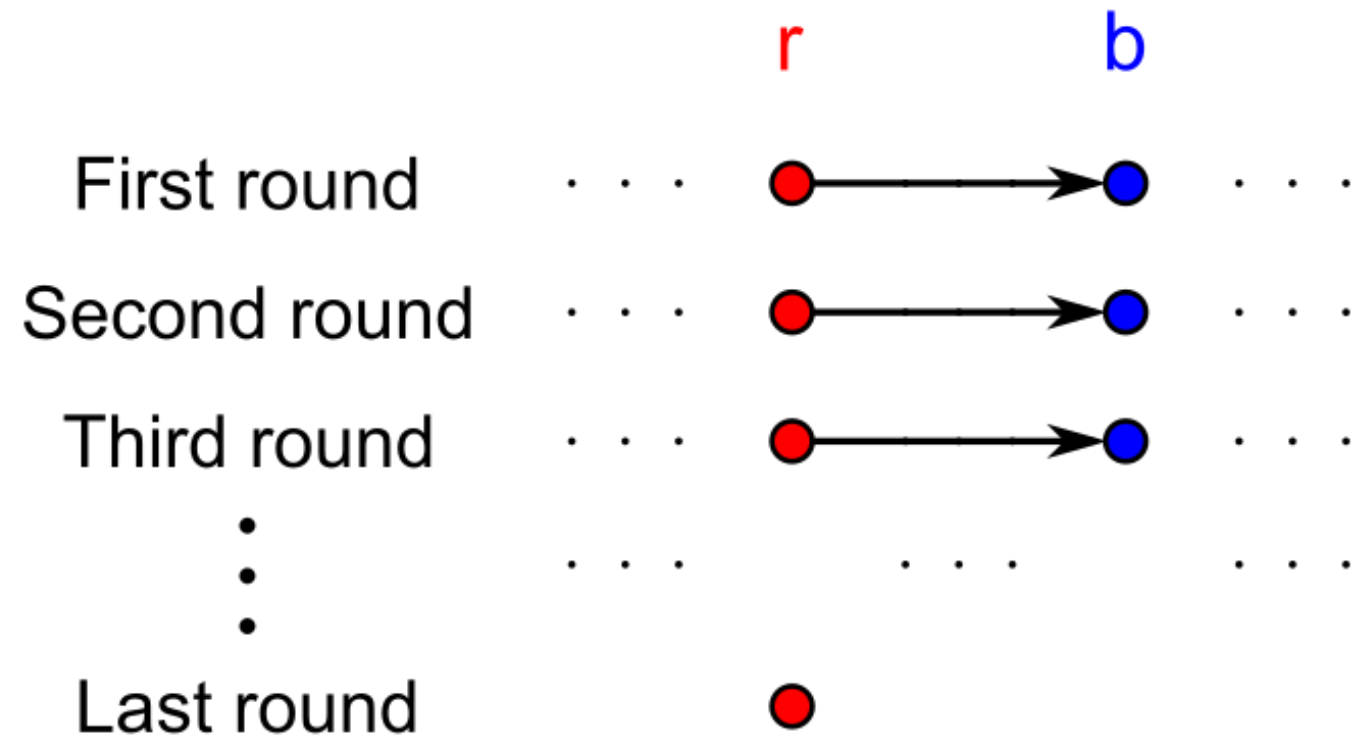
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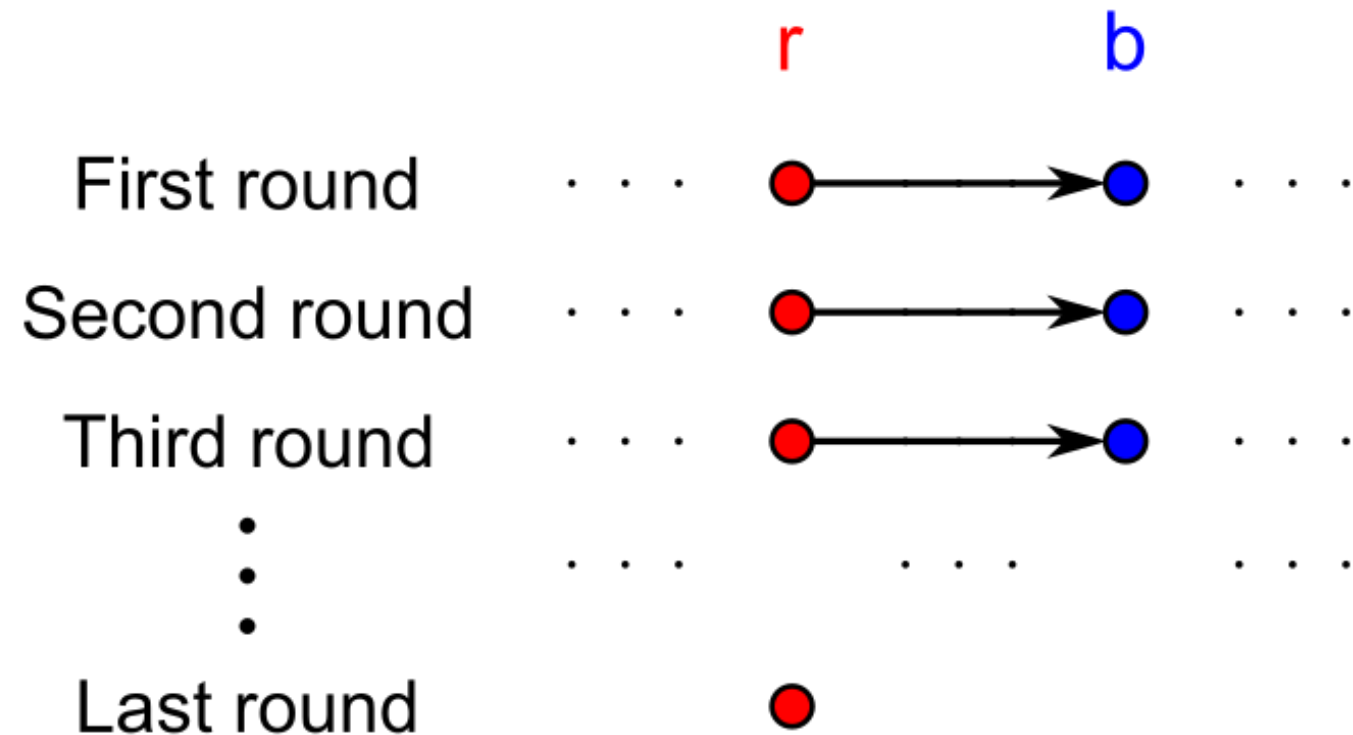
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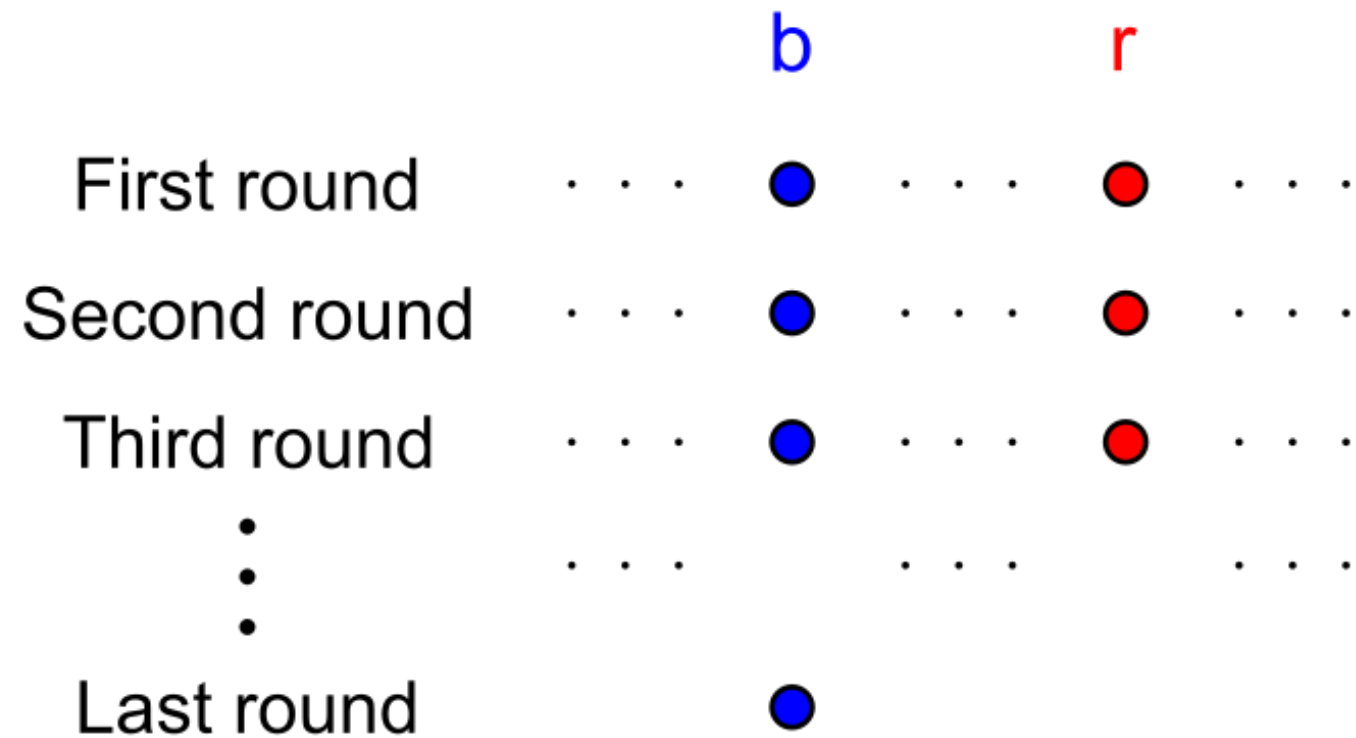
If r precedes b : Then, by additivity, $v_r(A_r) \geq v_r(A_b)$.

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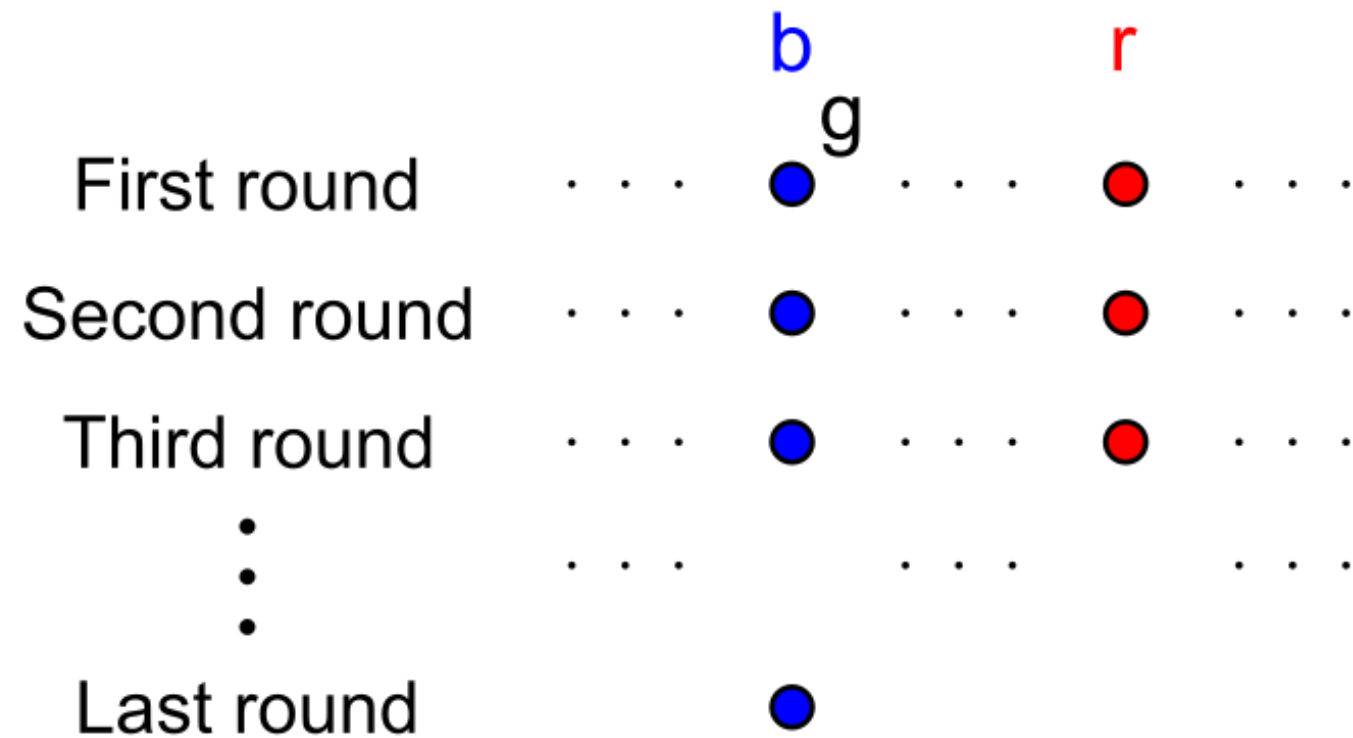
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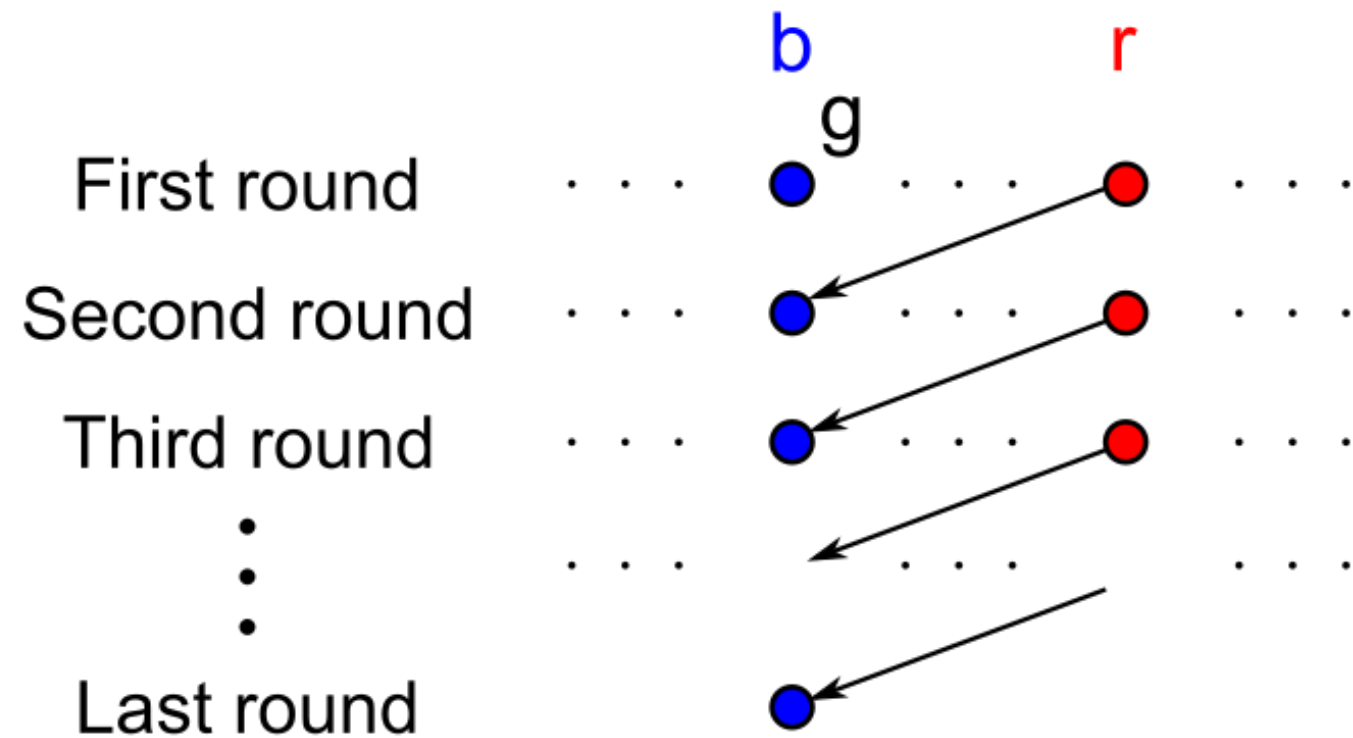
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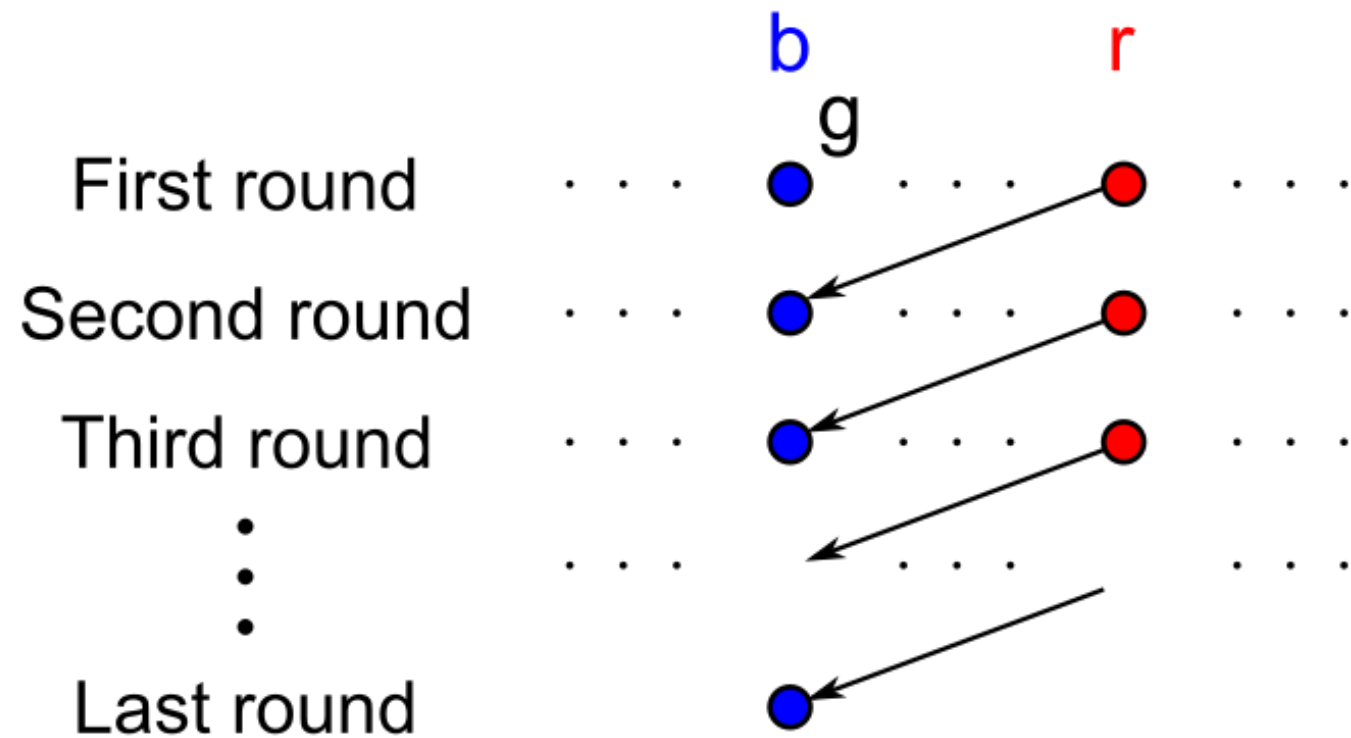
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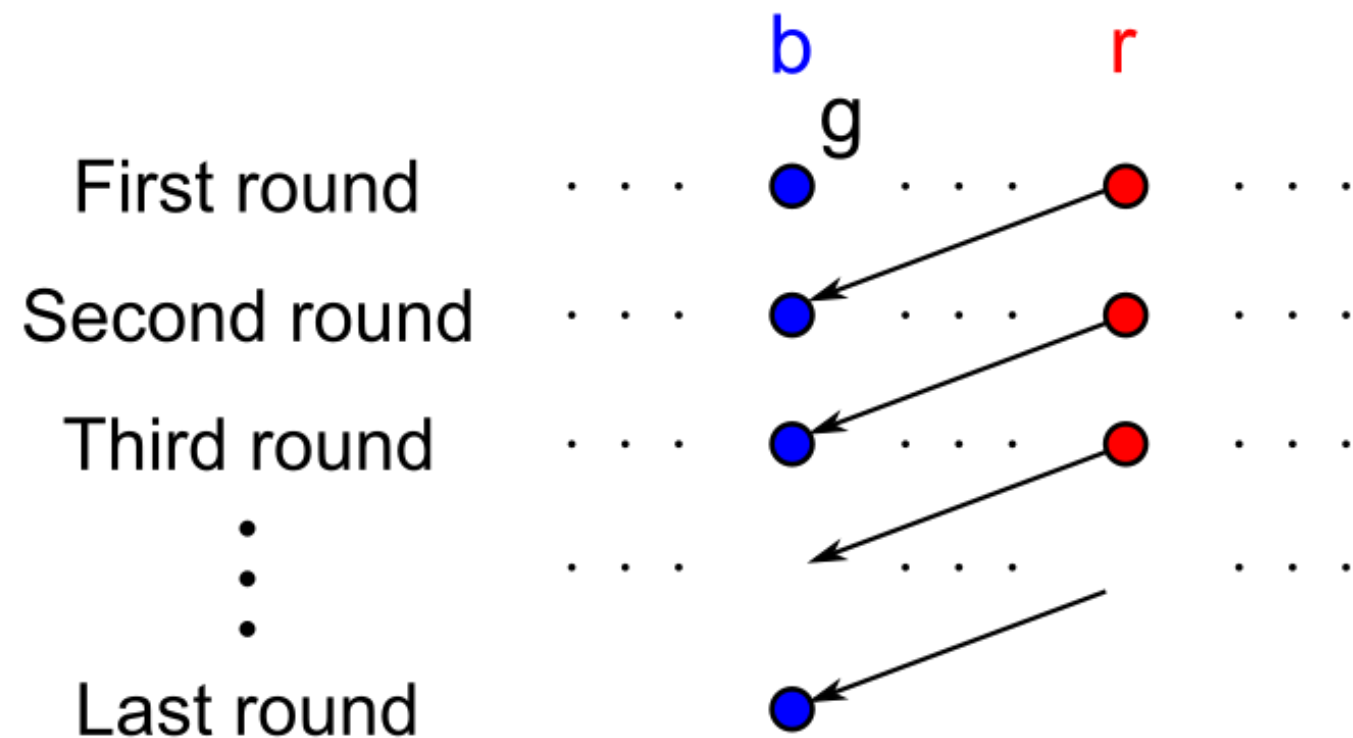
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Envy graph of an allocation

Envy graph of an allocation

- Vertices = agents
- Edge from vertex i to vertex k if agent i envies agent k in the given allocation.




Envy graph of an allocation

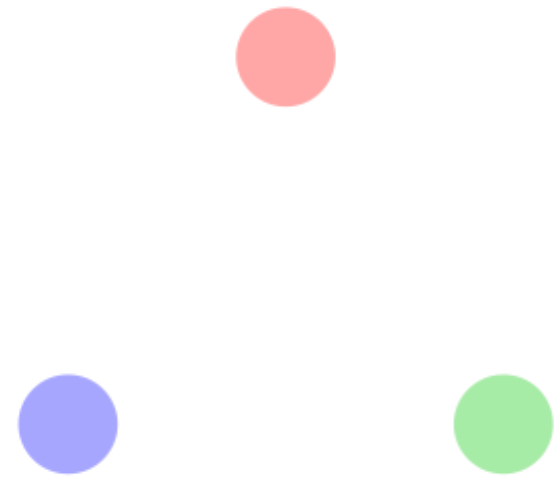
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	(A)	(B)	(C)	(D)	(E)
	4	1	2	4	2
	1	0	5	1	1
	1	1	5	1	1

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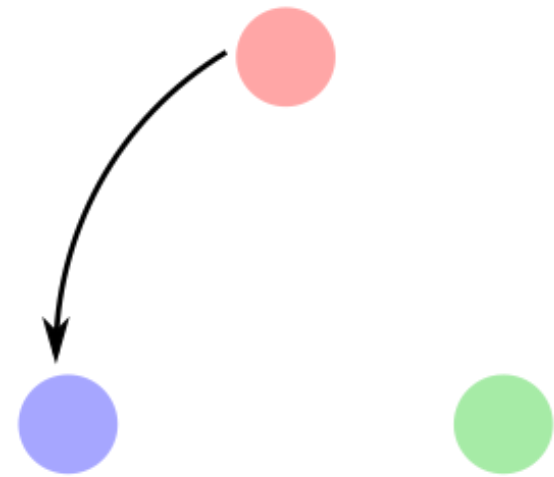
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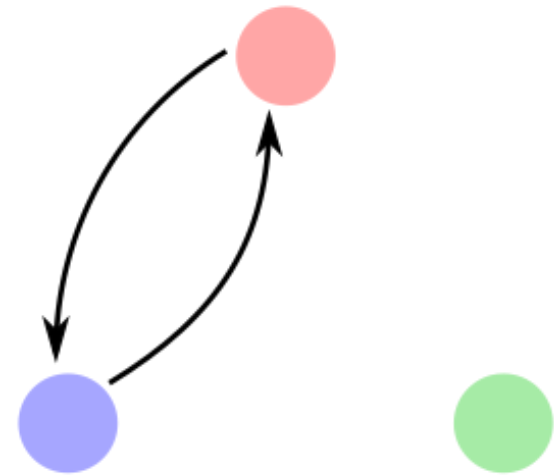
	(A)	(B)	(C)	(D)	(E)
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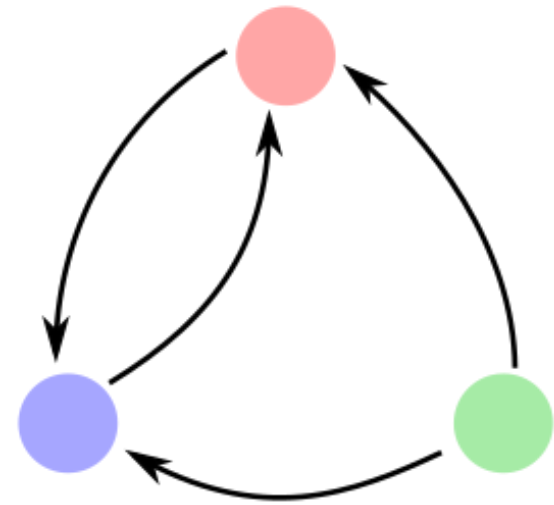
	(A)	(B)	(C)	(D)	(E)
	4	1	2	4	2
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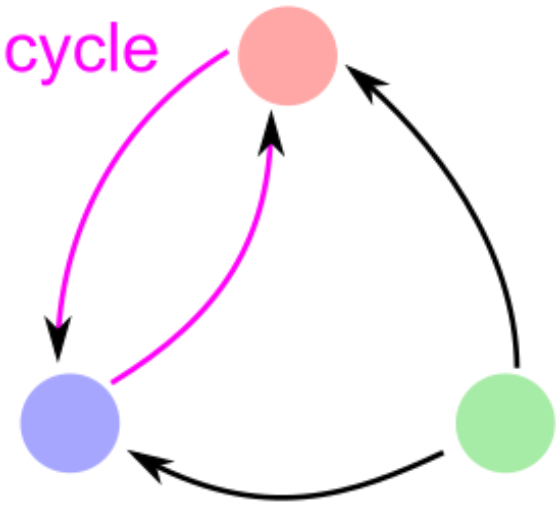


Envy graph of an allocation

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Envy cycle



Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

While there is an unallocated good

Envy-cycle elimination algorithm

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While there is an unallocated good

- If the envy graph has a source vertex, assign the good to that agent.

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Envy-cycle elimination algorithm

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- If the envy graph has a source vertex, assign the good to that agent.
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each agent in the cycle gets the bundle
that it is pointing to

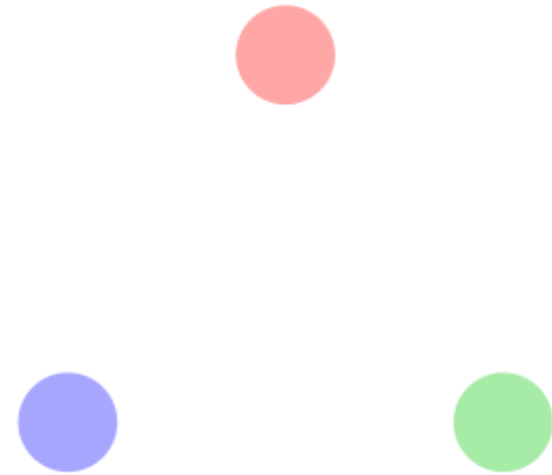
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	(A)	(B)	(C)	(D)
	0	2	0	1
	1	2	5	10
	1	4	2	10



Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

While there is an unallocated good

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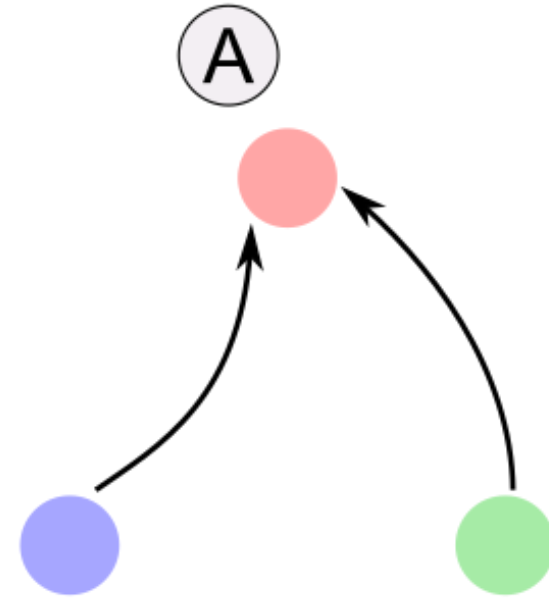
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	(A)	(B)	(C)	(D)
	0	2	0	1
	1	2	5	10
	1	4	2	10






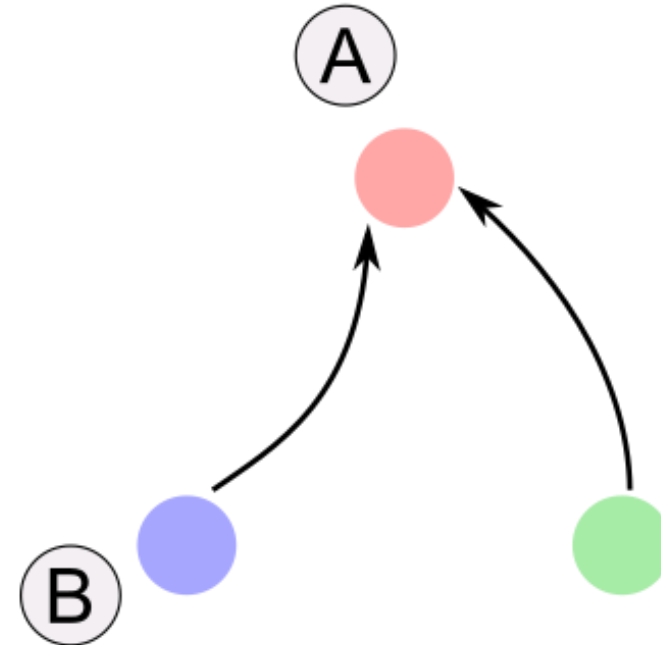
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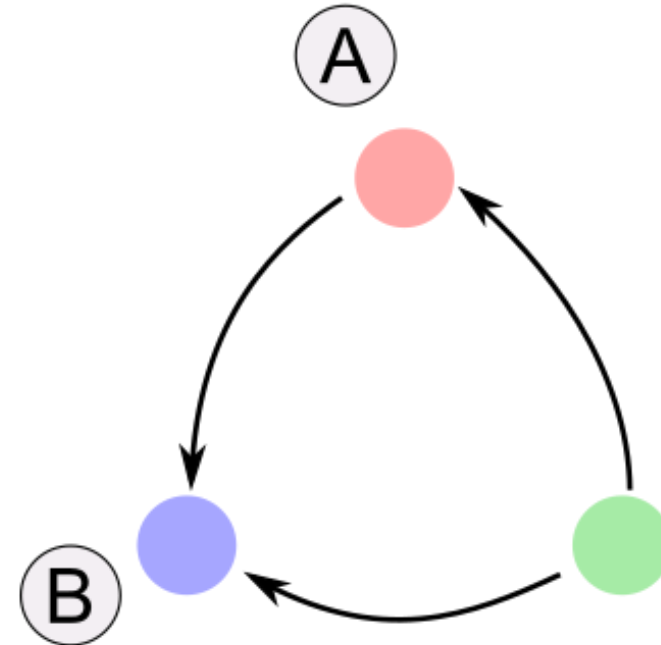
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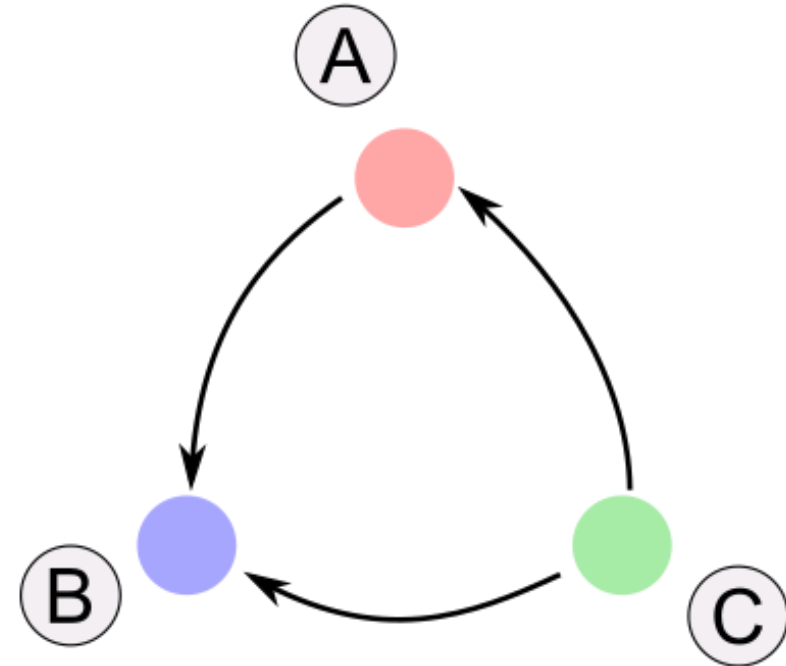
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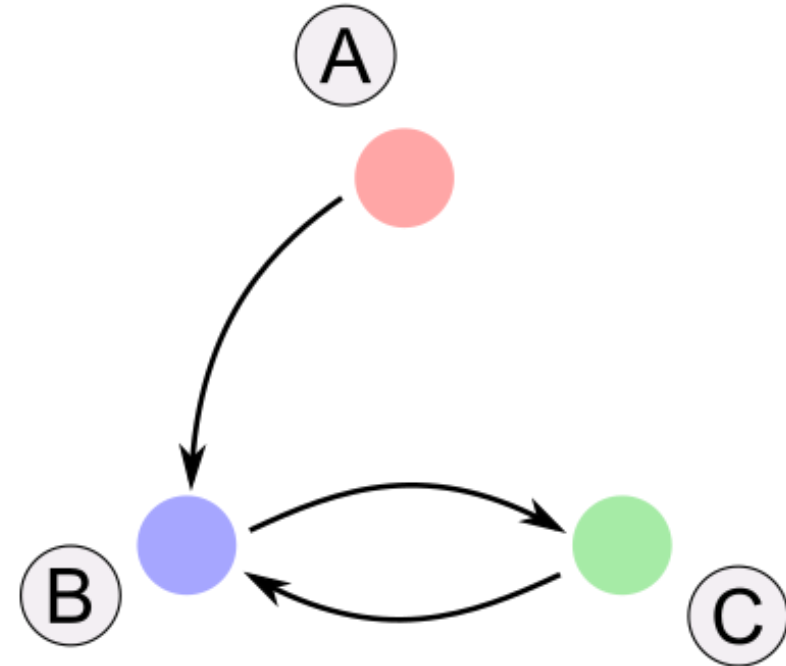
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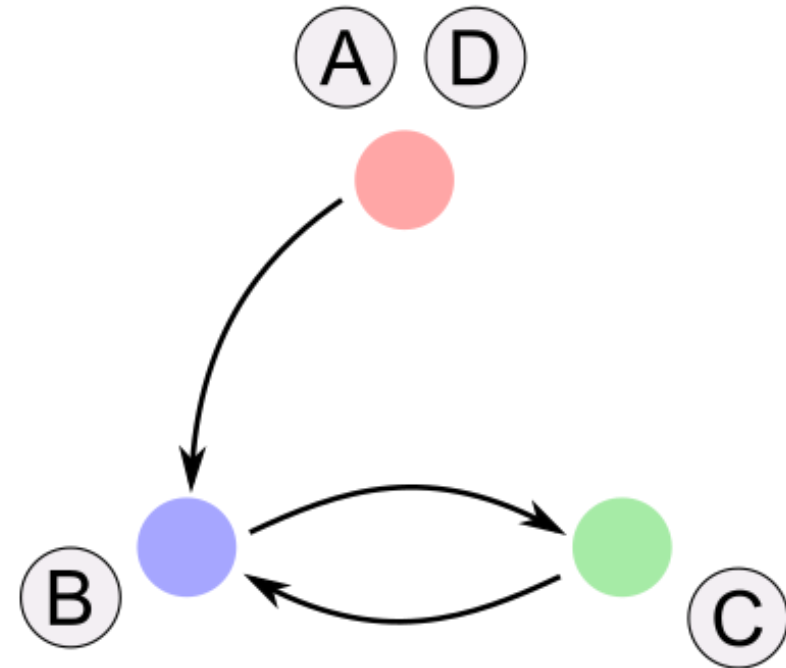
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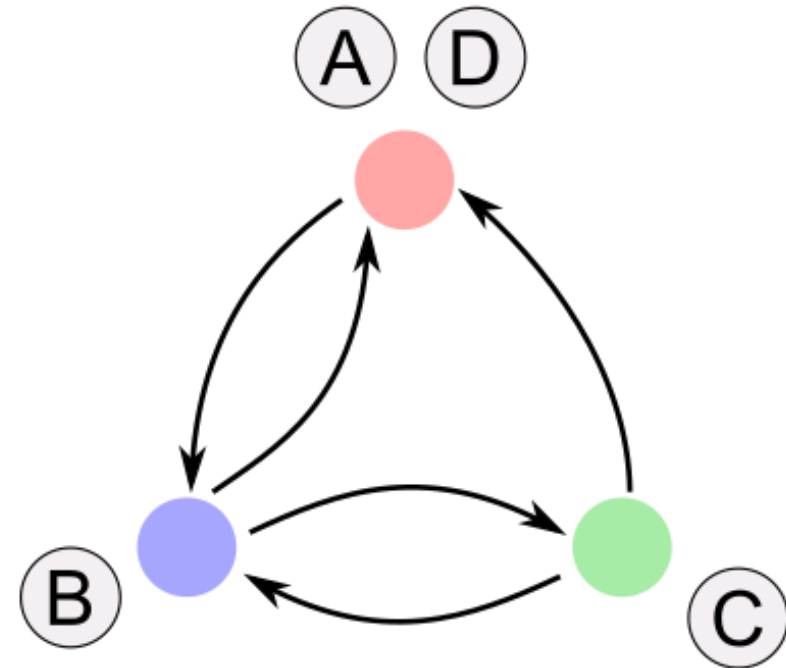
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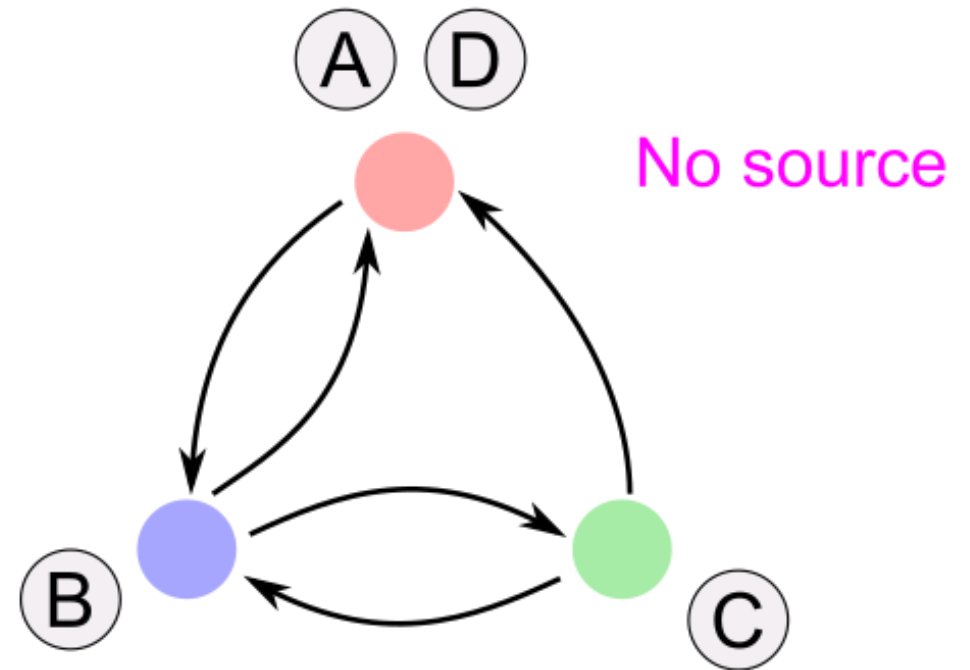
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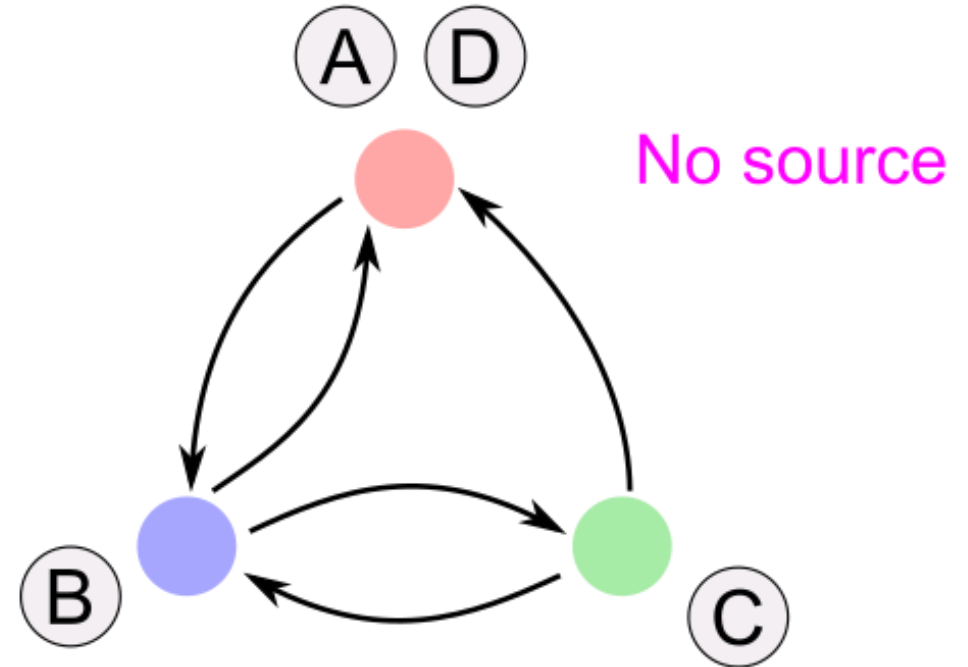
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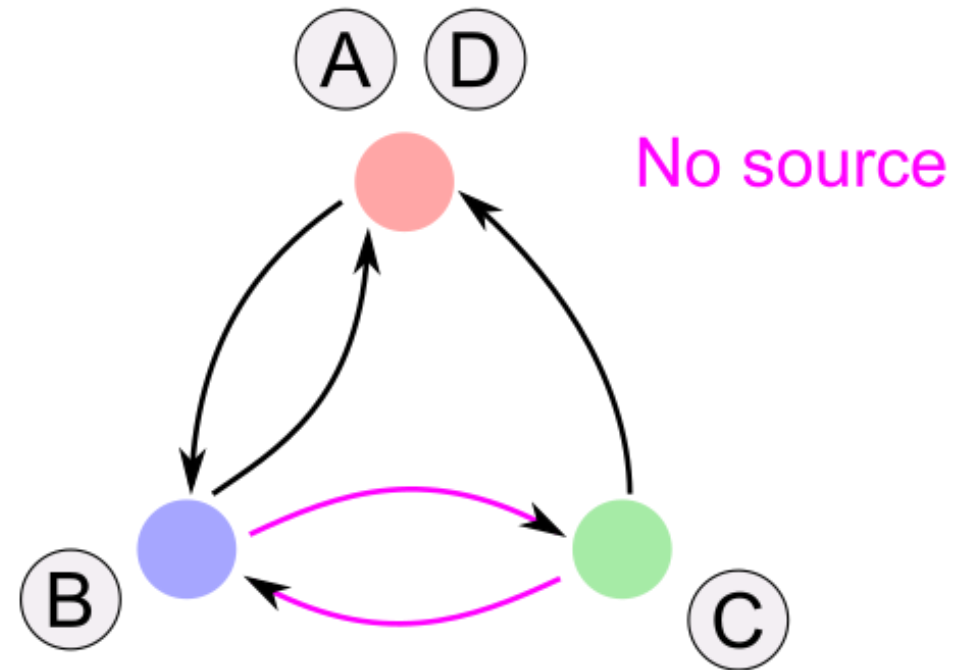
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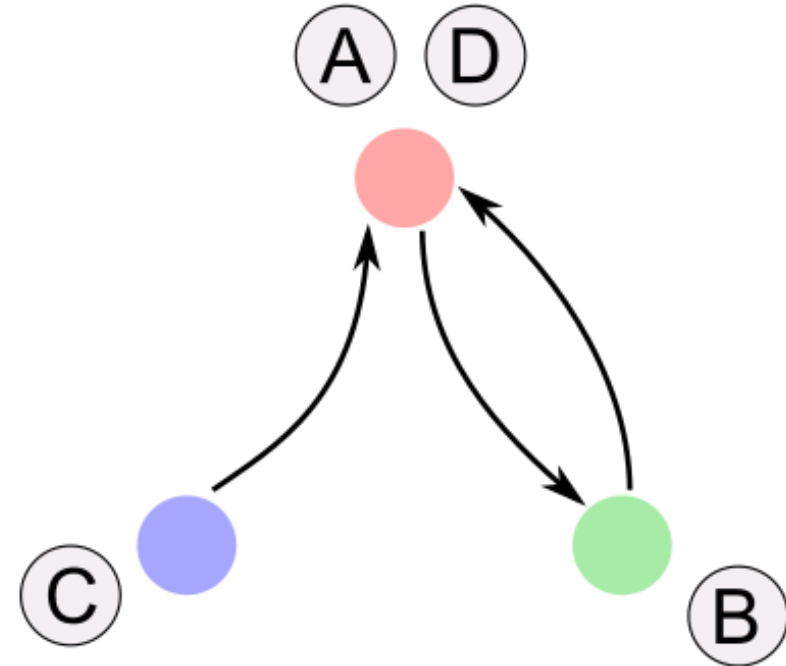
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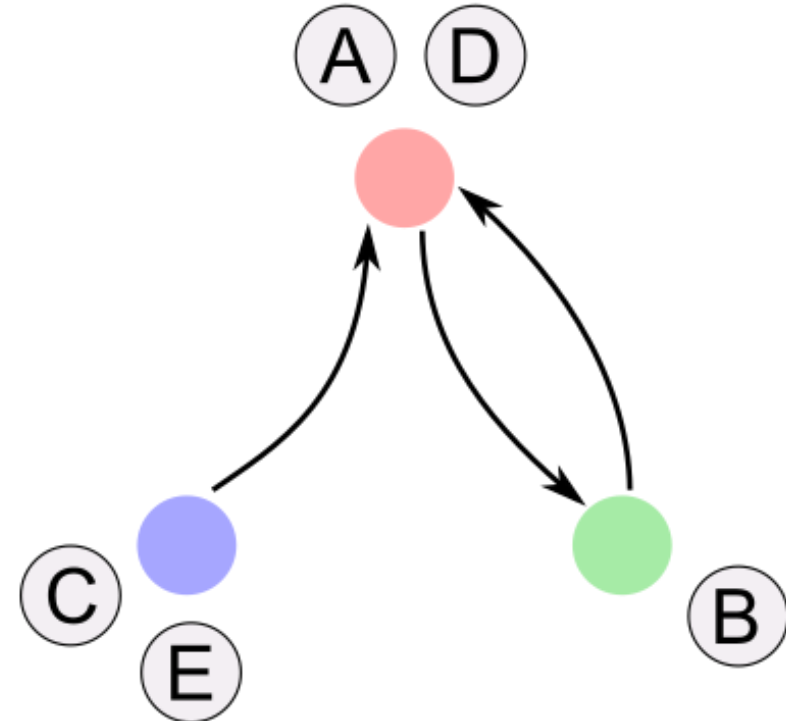
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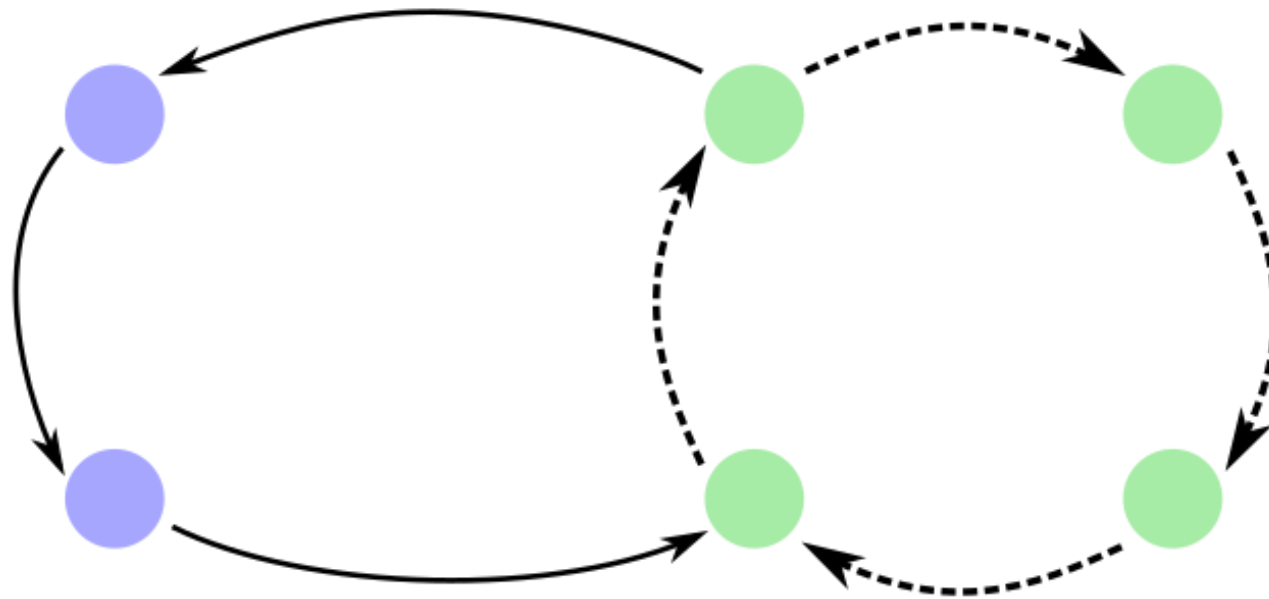
- With n agents, at most $O(n^2)$ cycle resolutions required to create a source.
- Polynomial running time!

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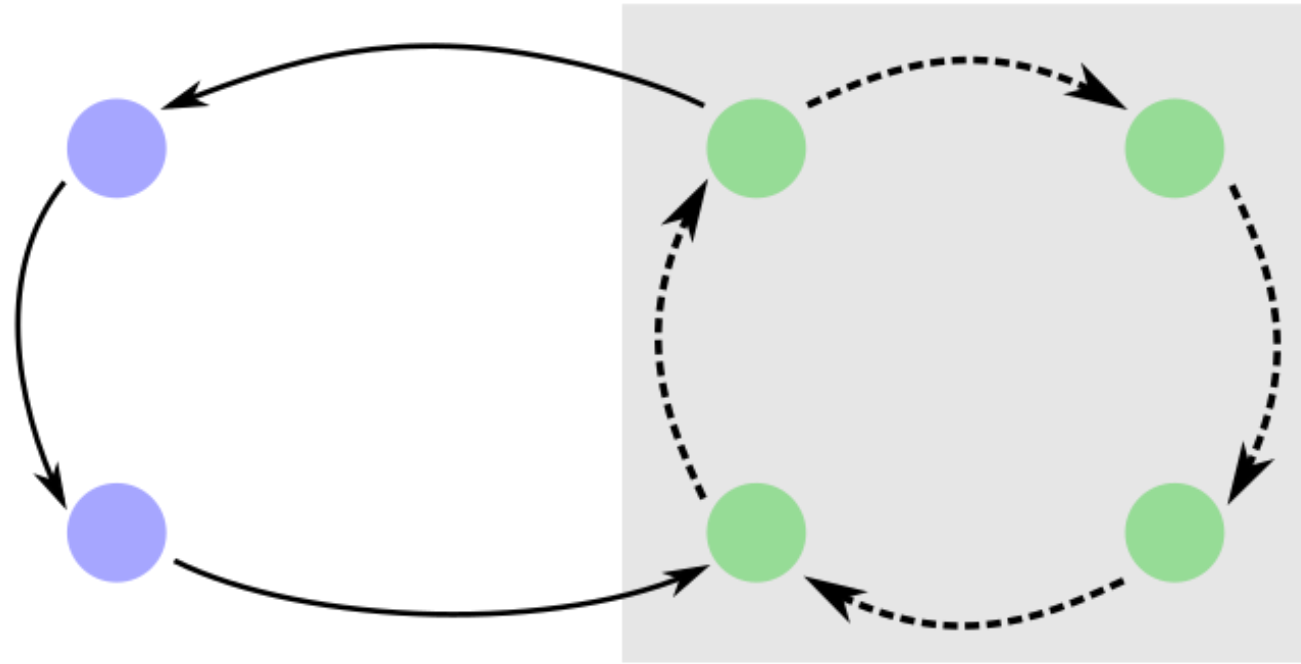
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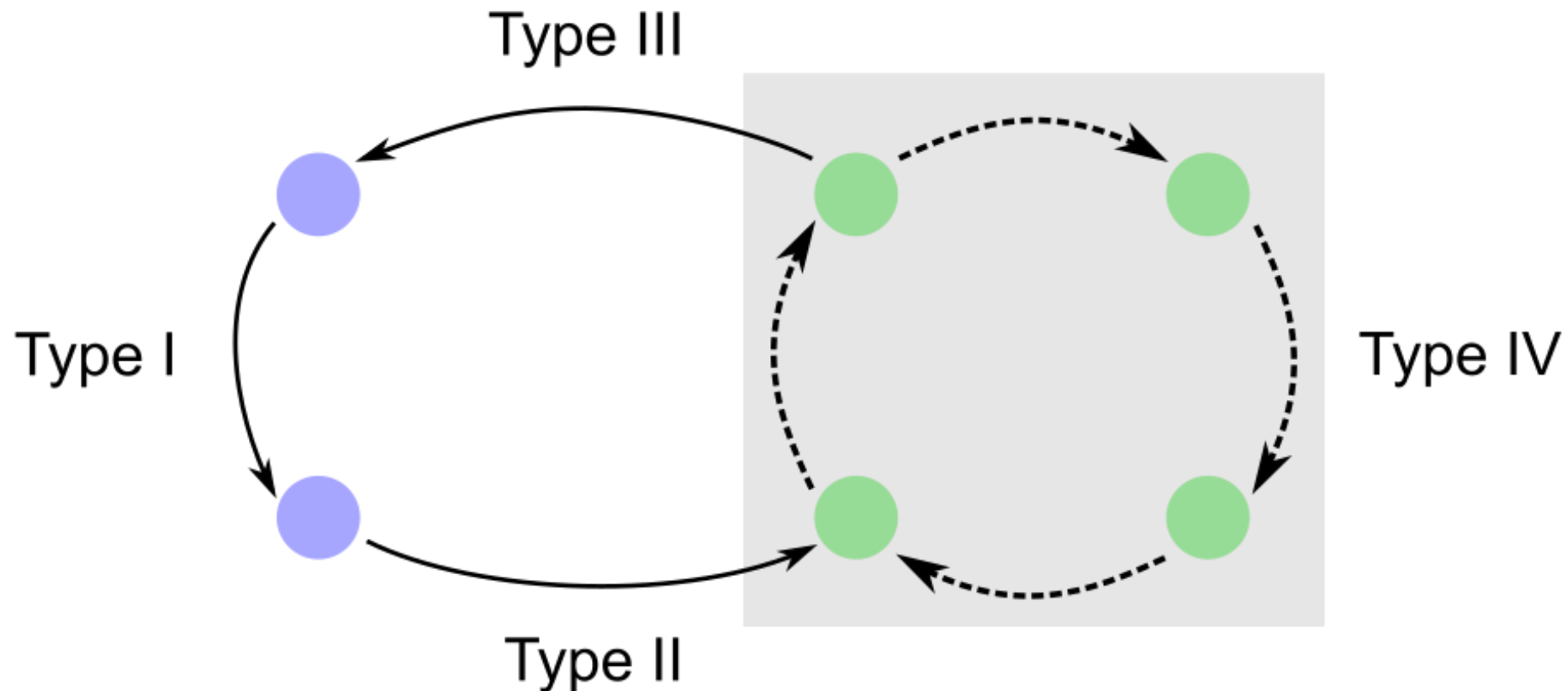
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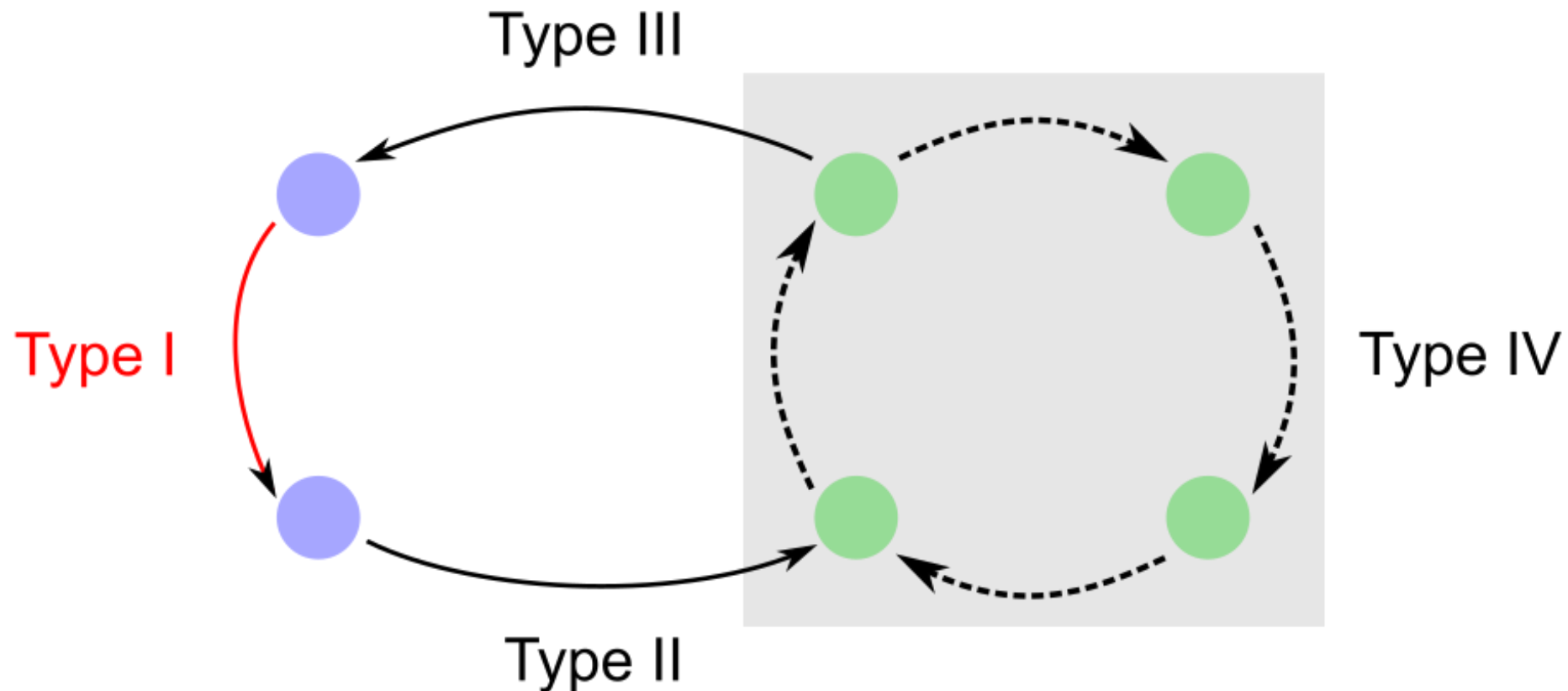
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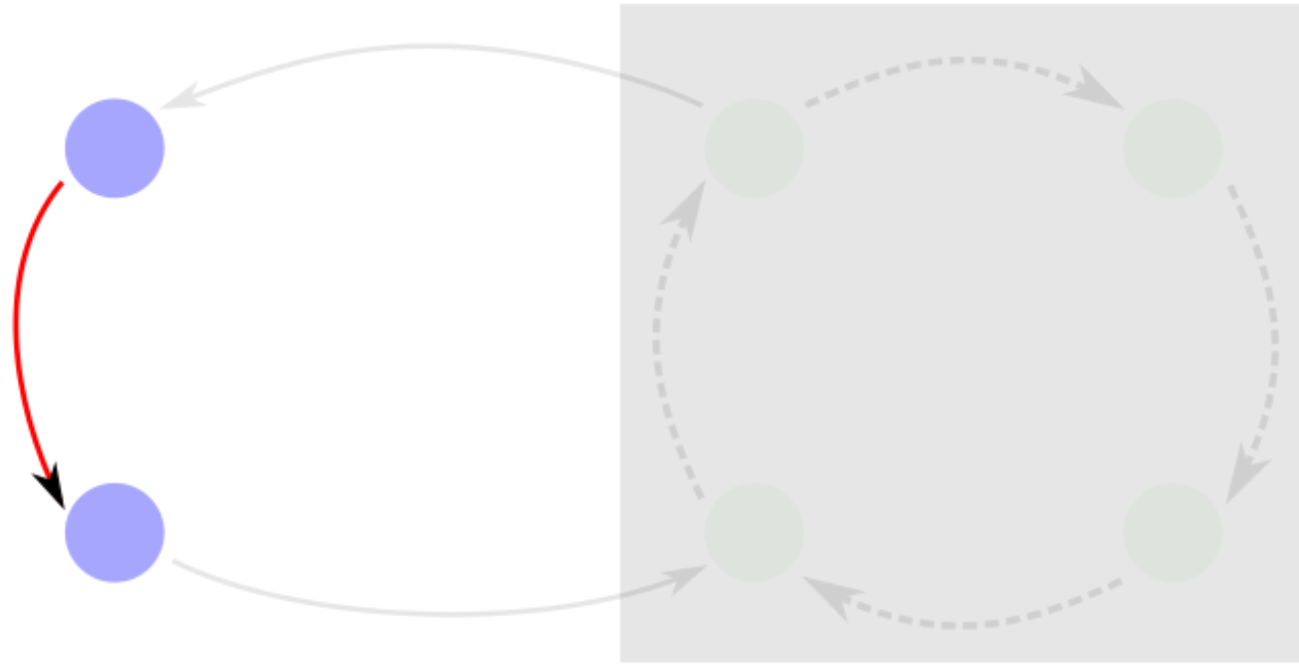
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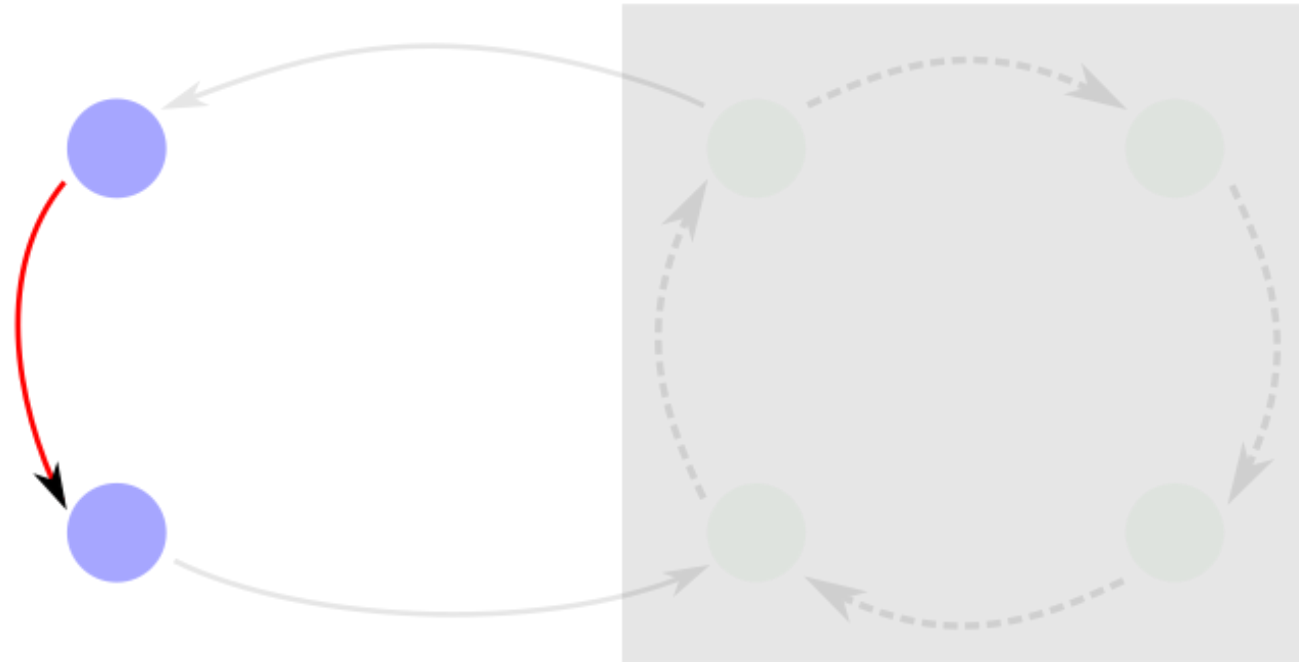
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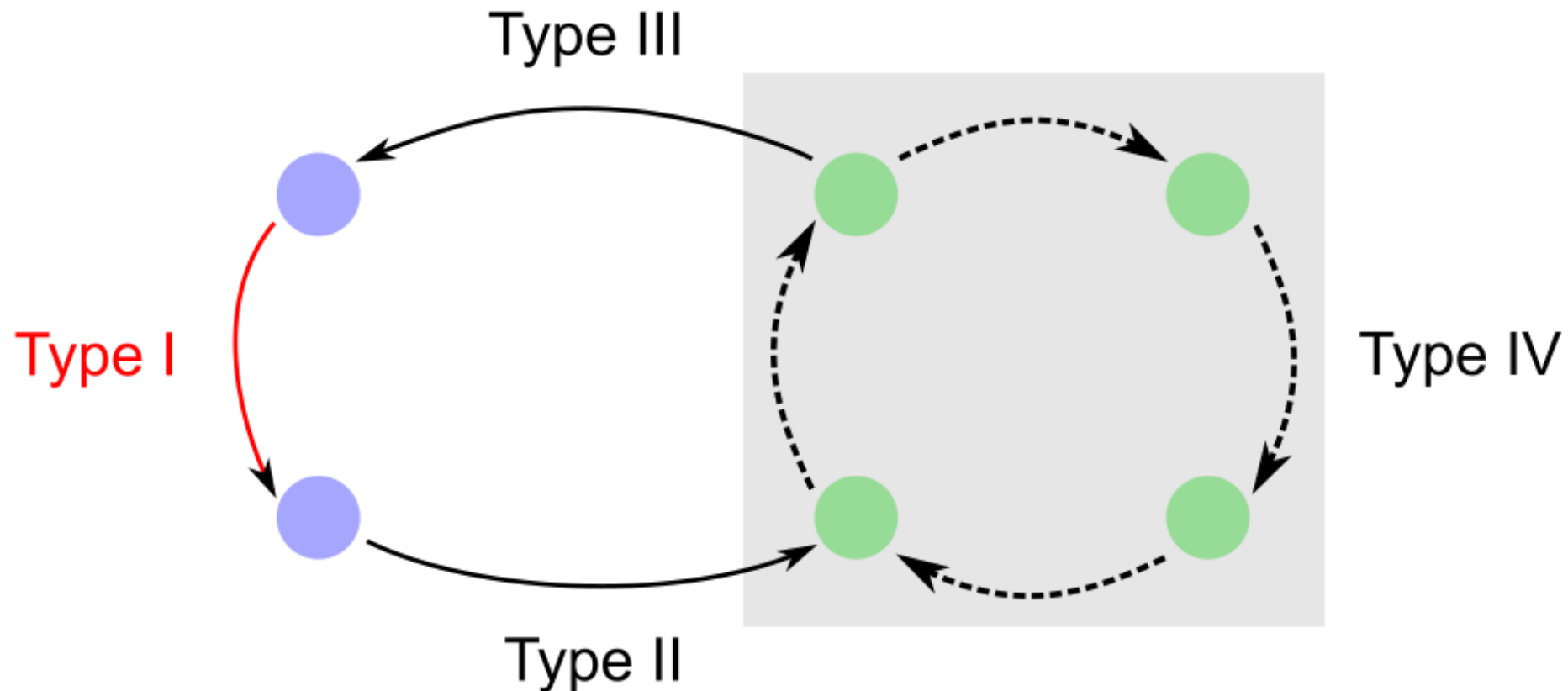
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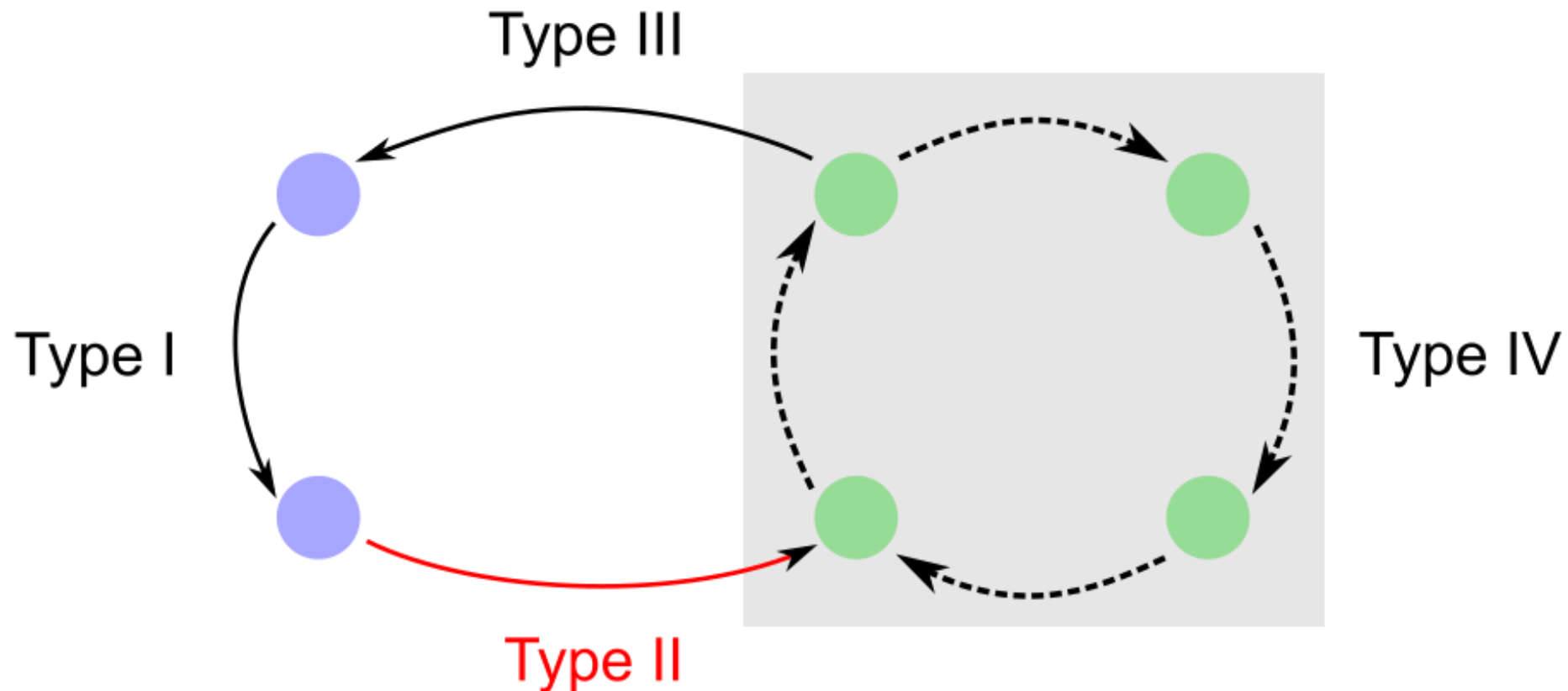
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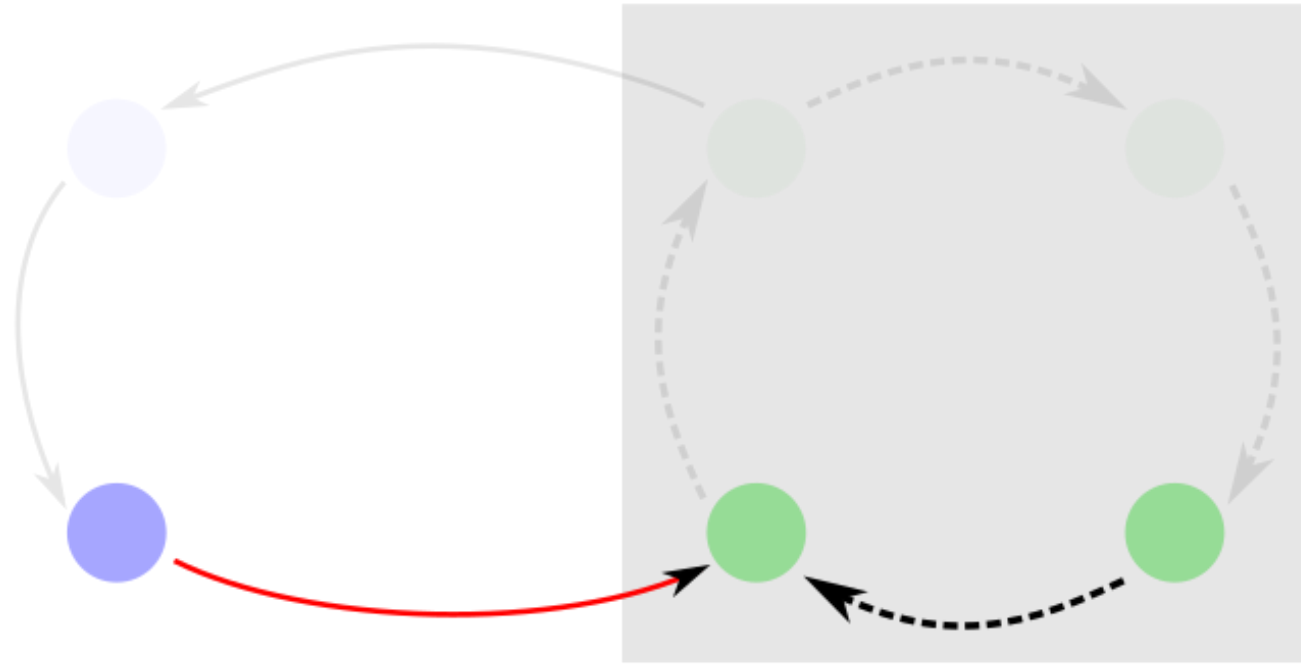
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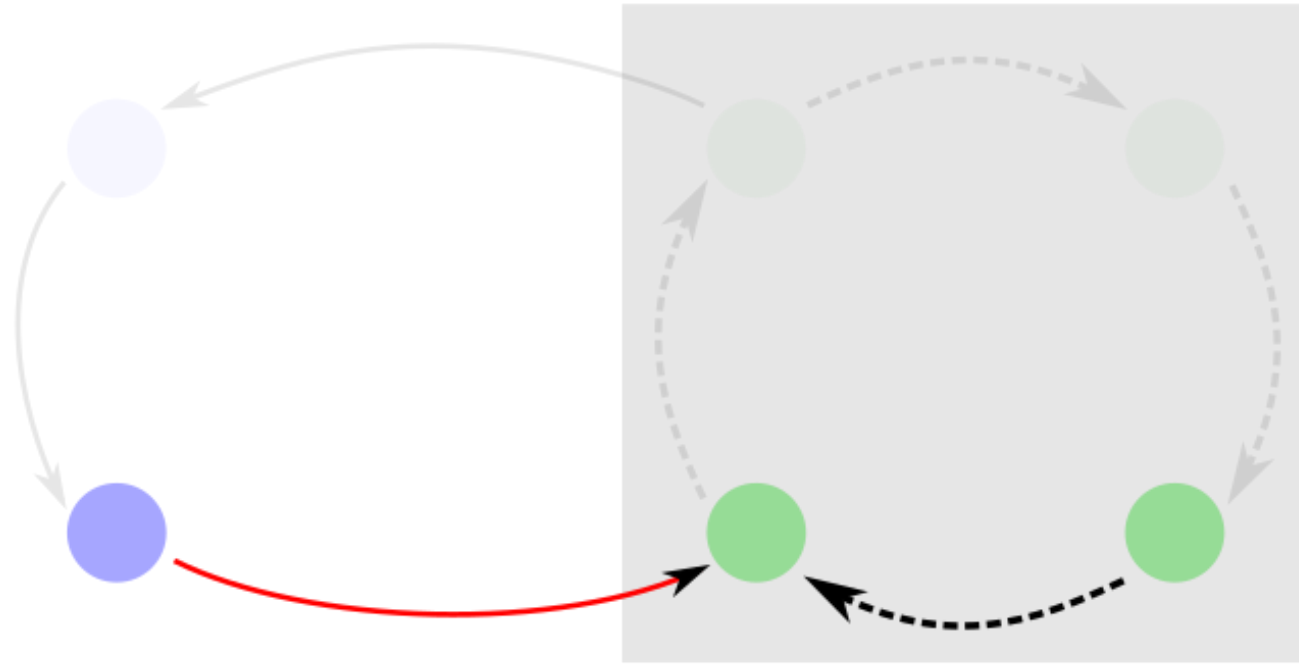
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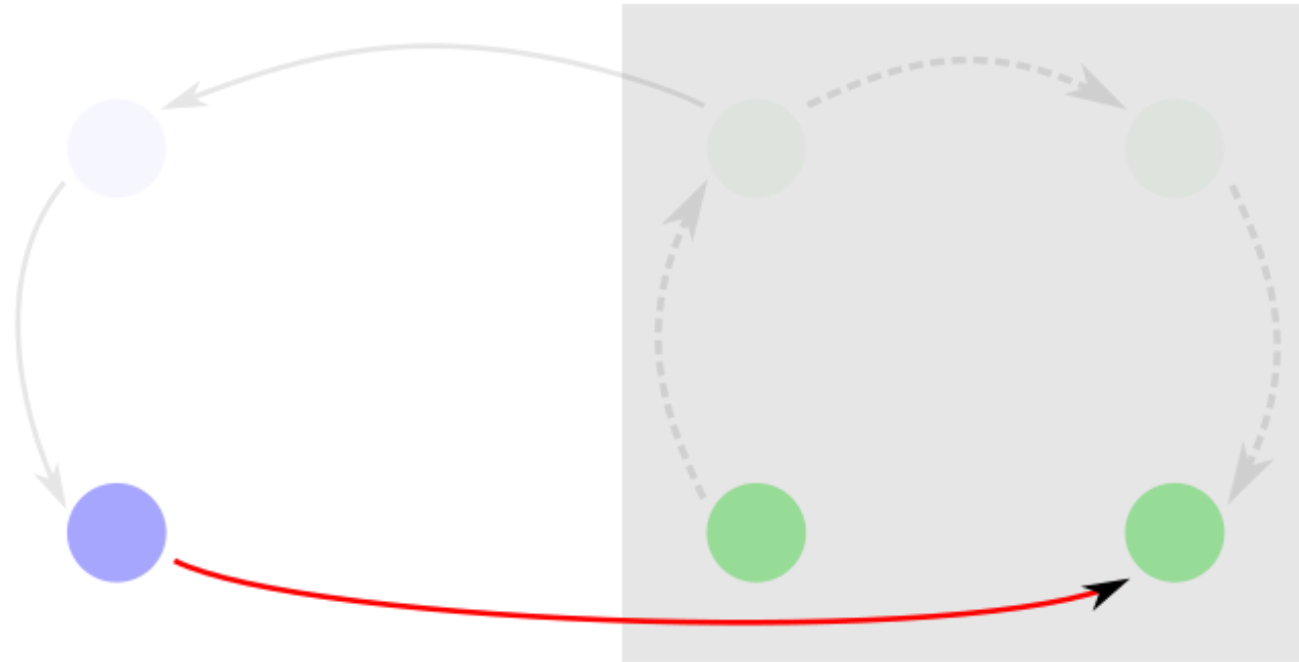
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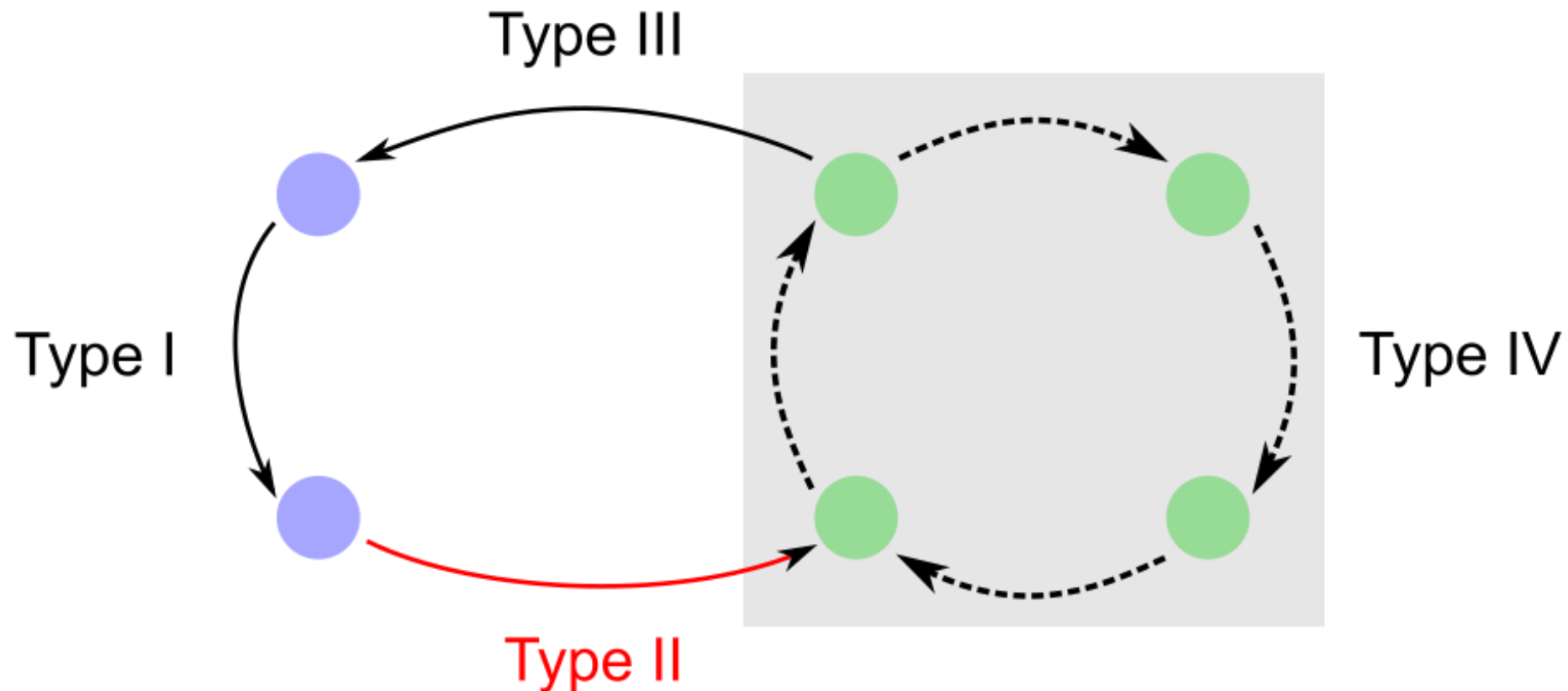
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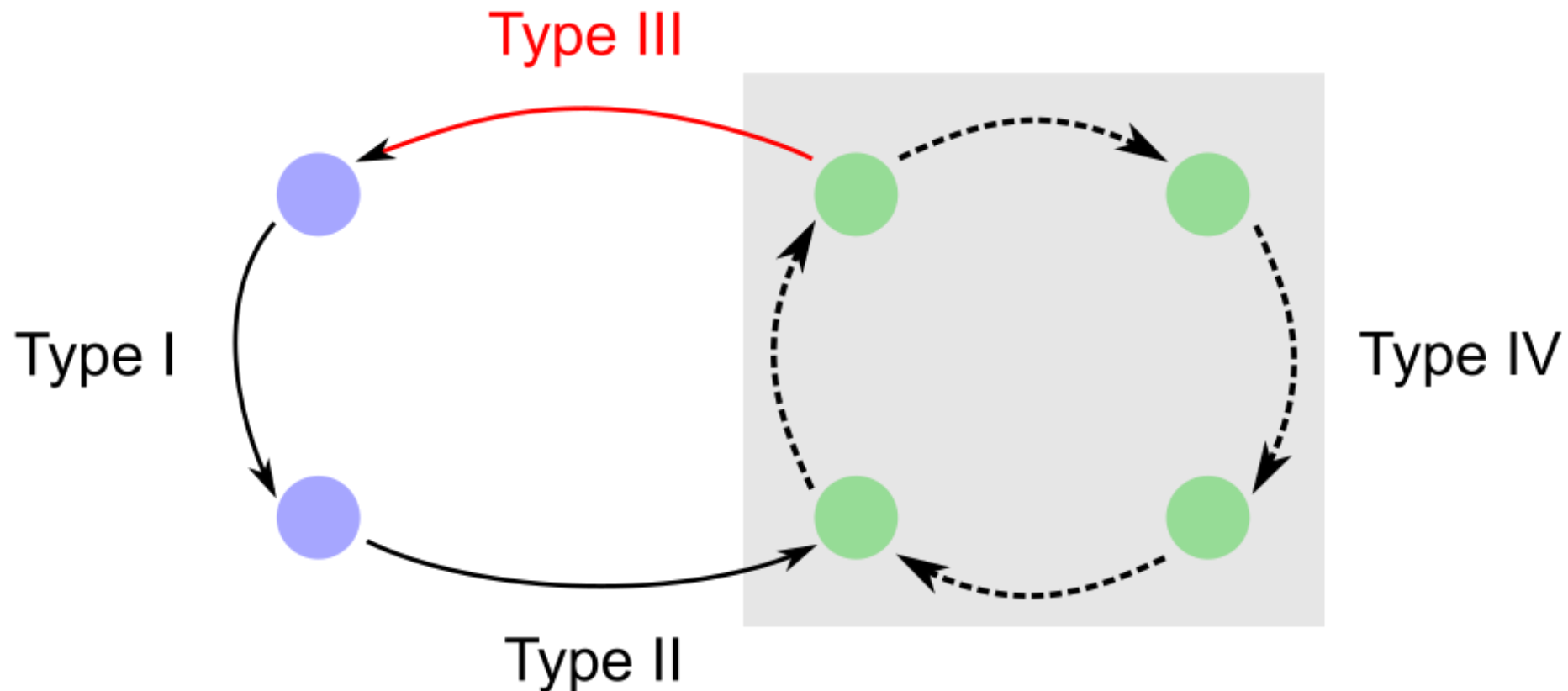
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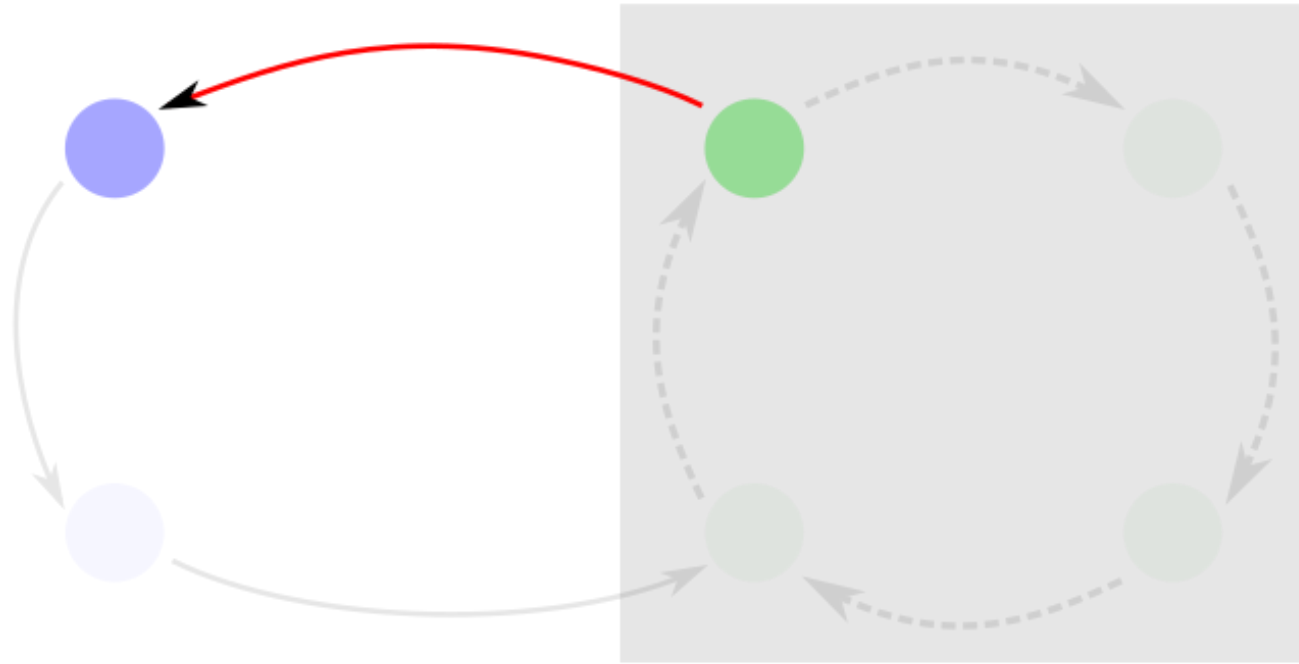
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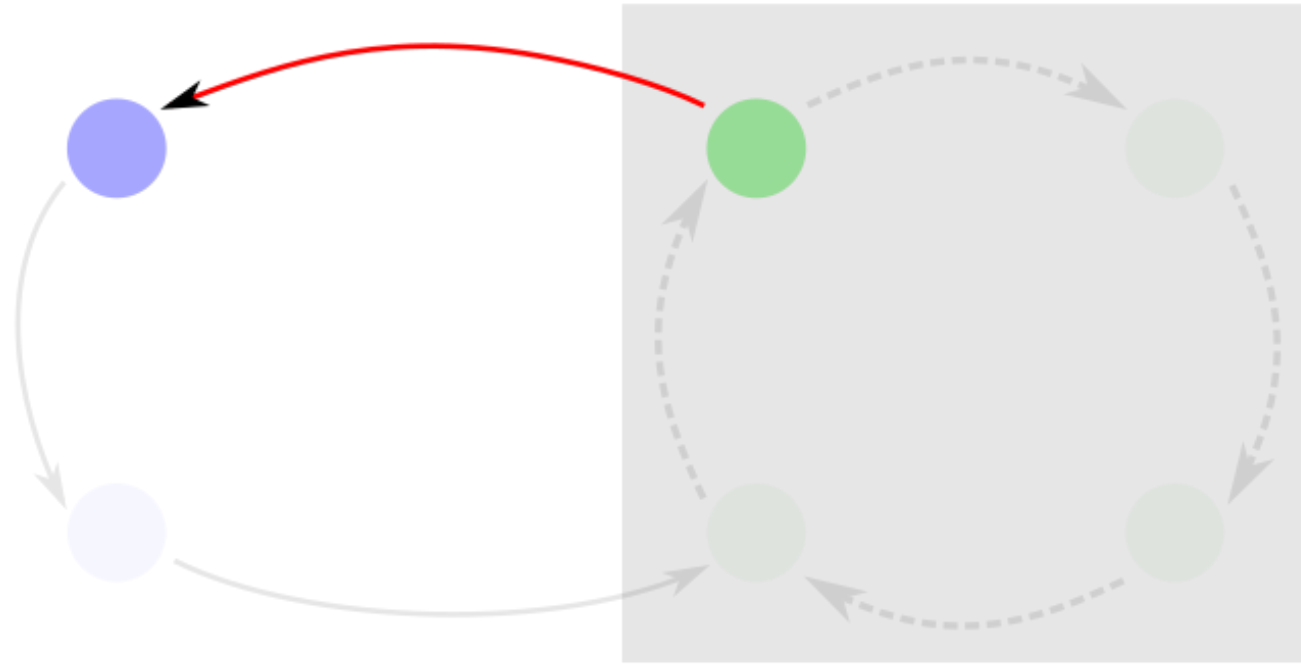
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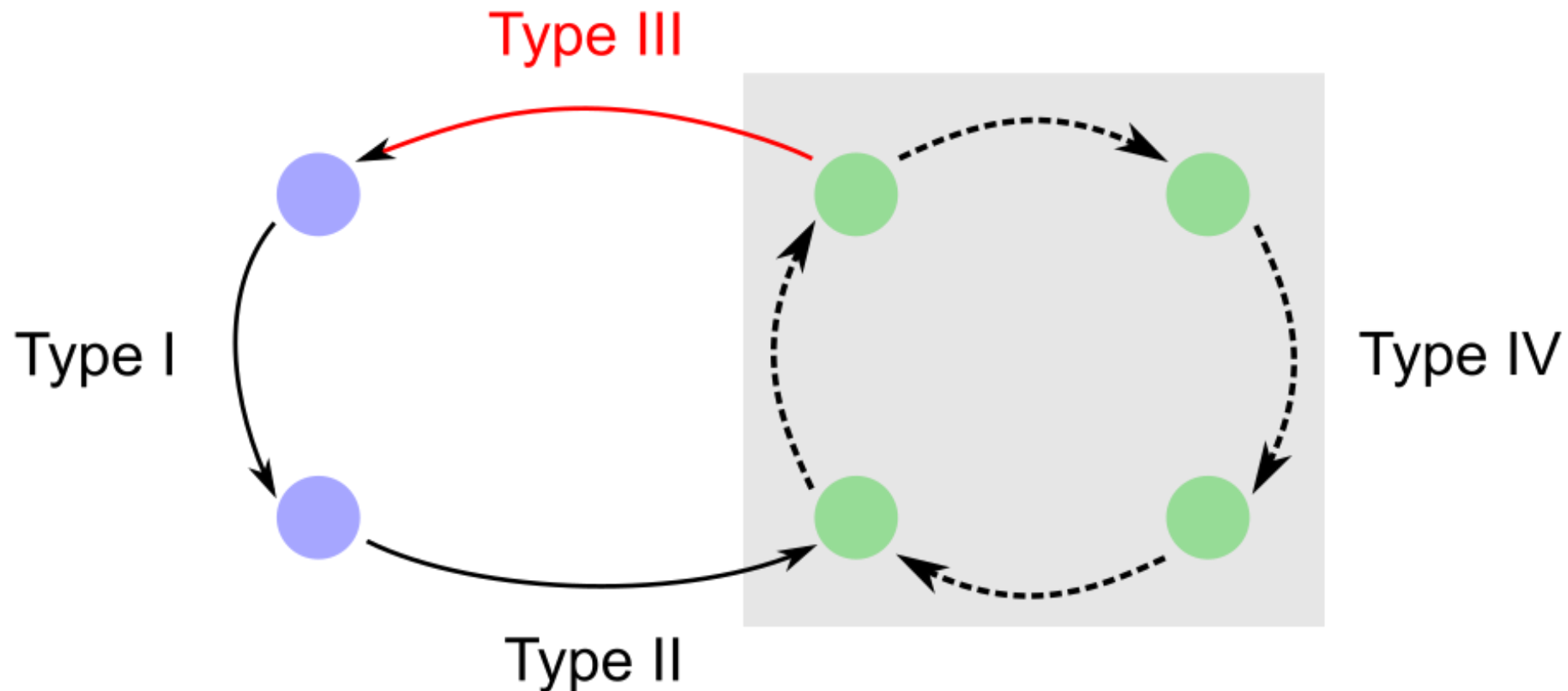
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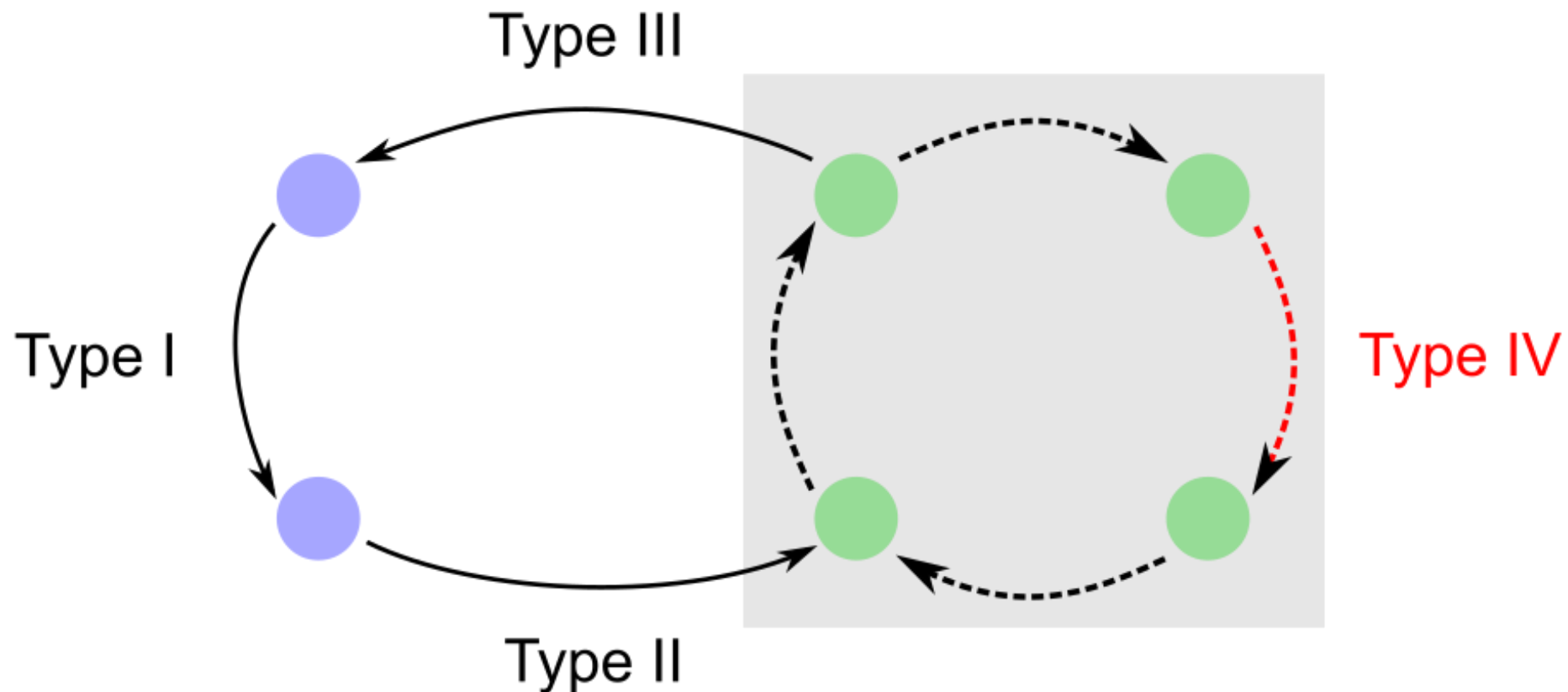
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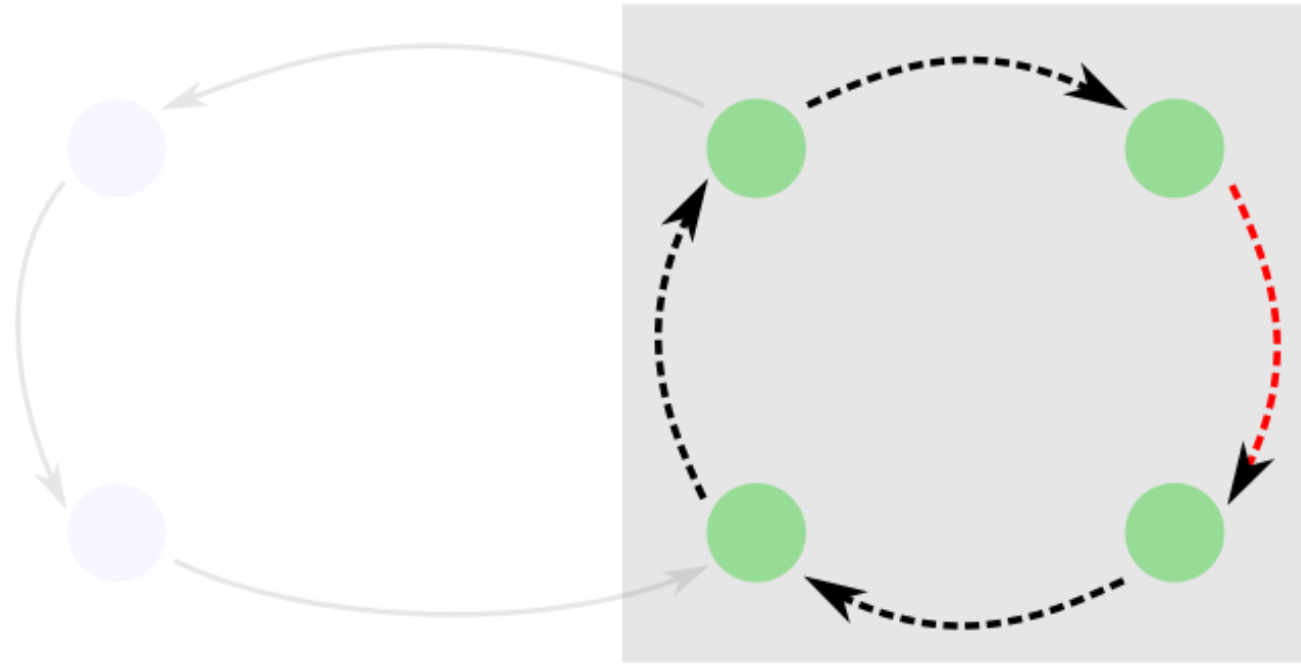
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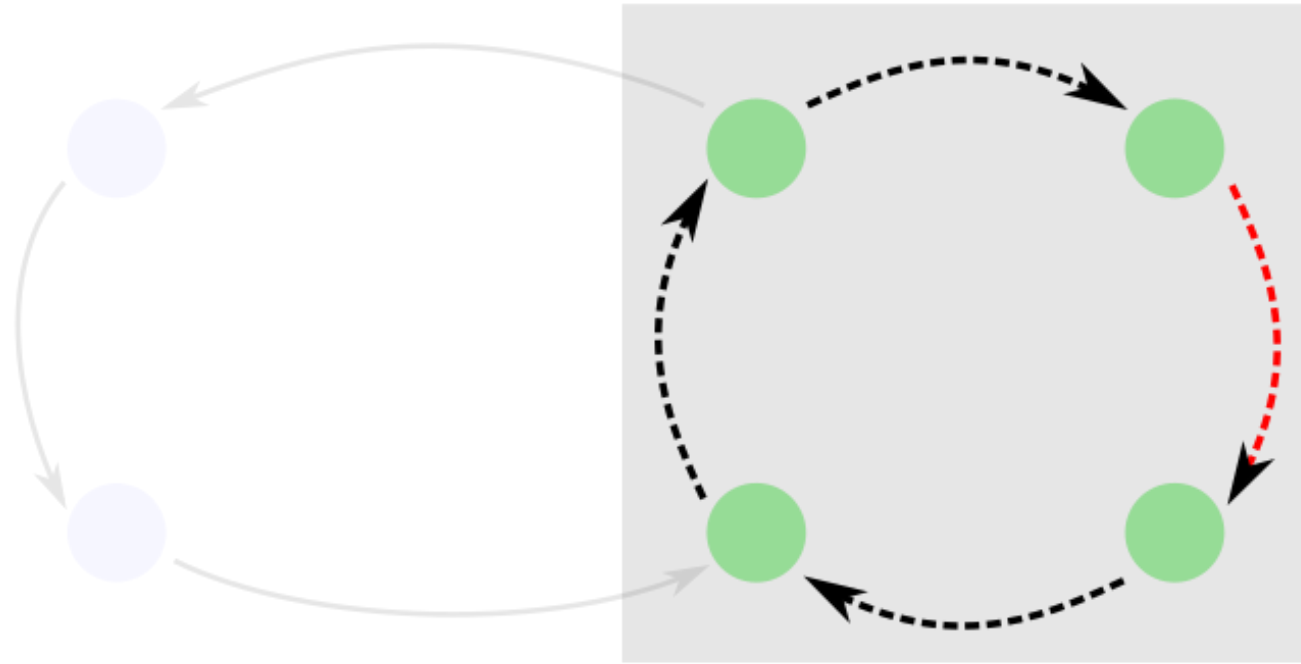
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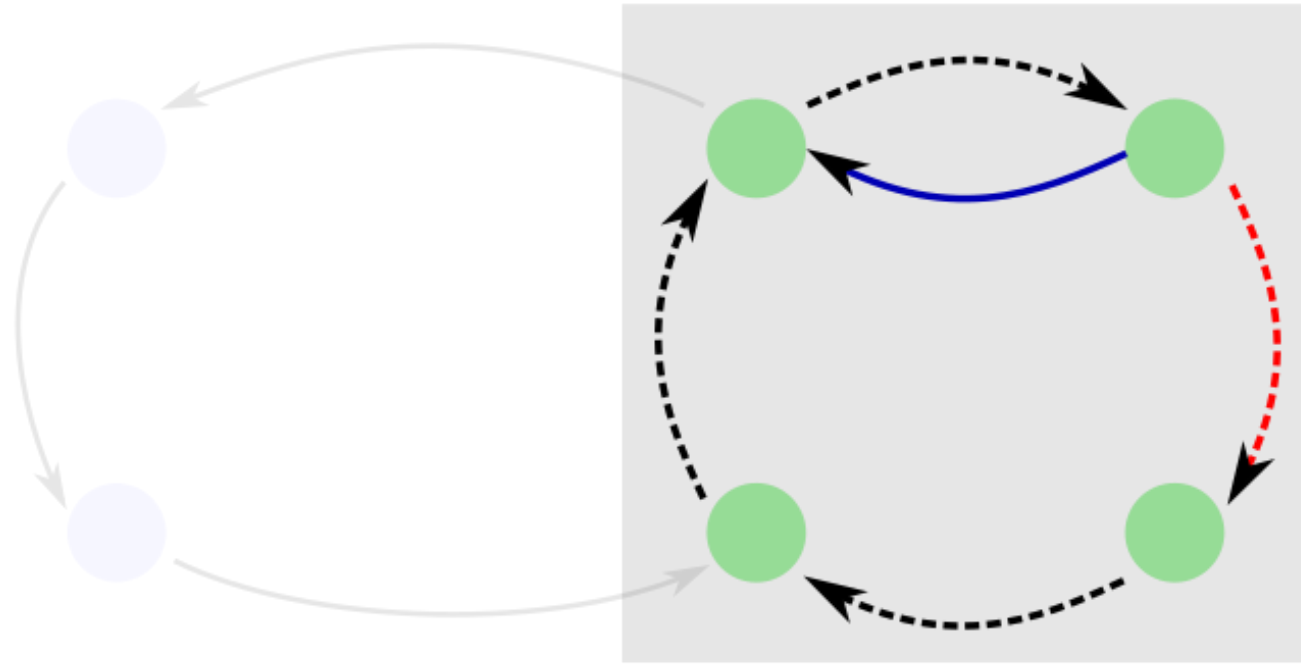
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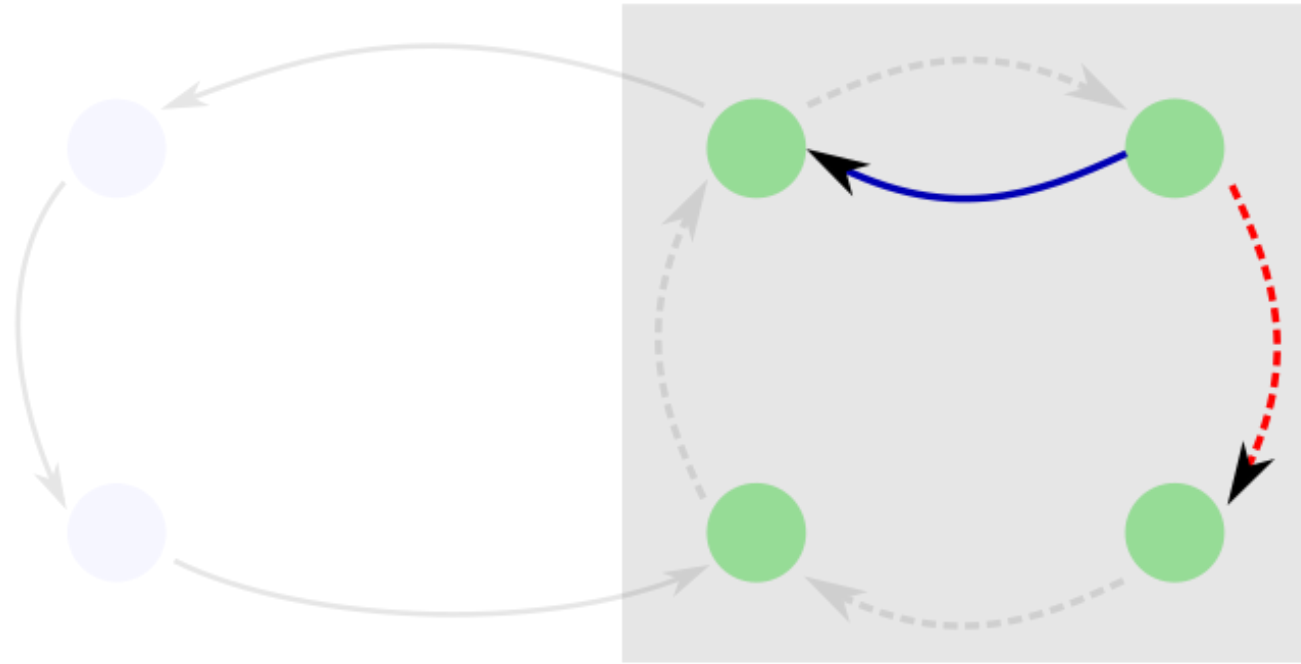
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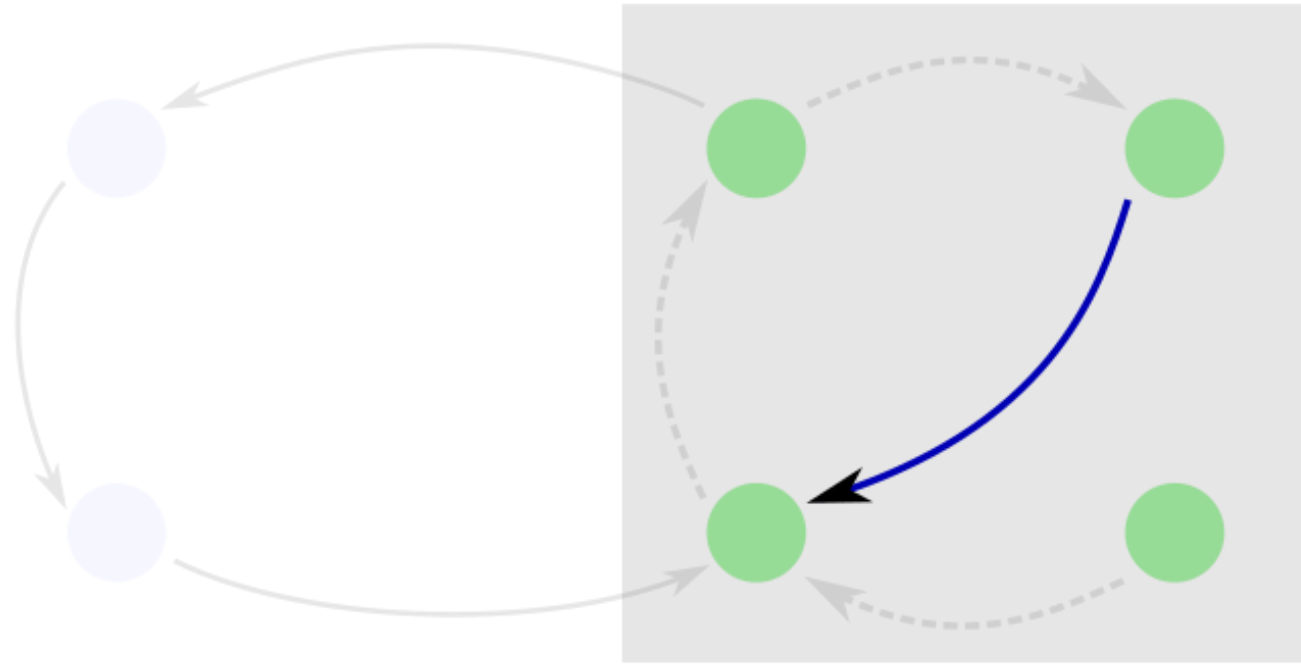
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Allocation A is EF1 if for every pair of agents i, k , there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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We will argue that each iteration of the algorithm "preserves" EF1.

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If the partial allocation at the beginning of an iteration is EF1, then the partial allocation at the end of that iteration is also EF1.

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Suppose good g is assigned to the source agent s . Then,

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which means that EF1 is preserved.

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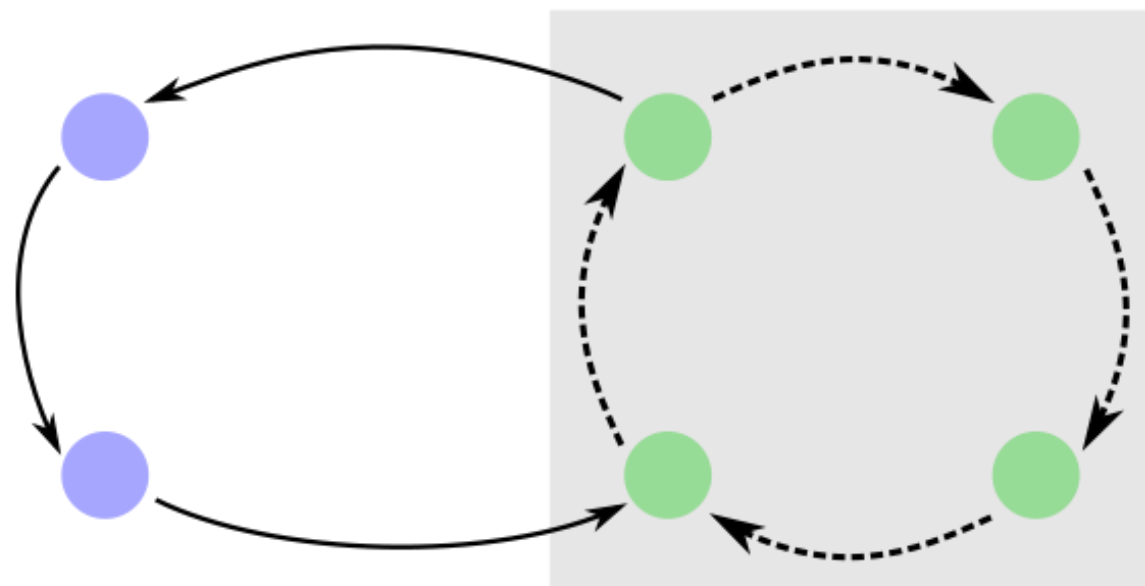
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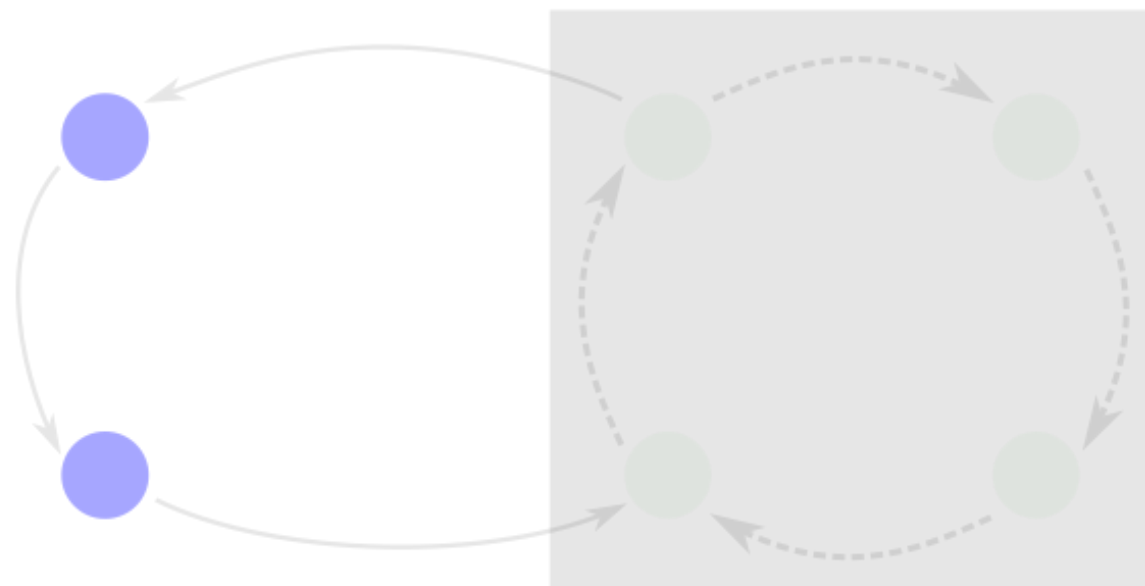


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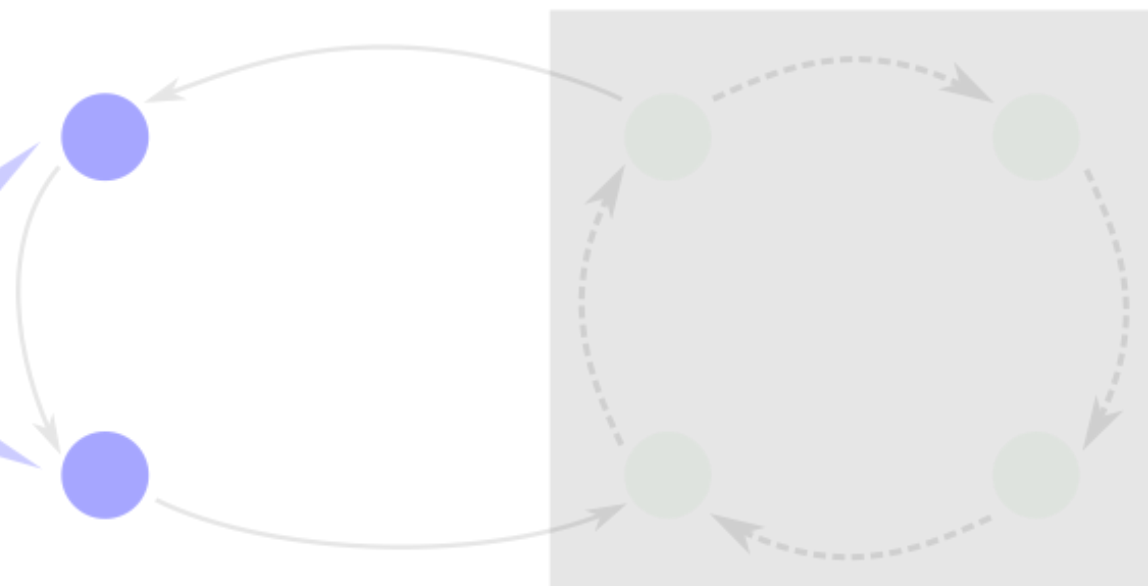
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From their perspective, the bundles in the cycle are only shifted around. So, EF1 relations are the same as before.

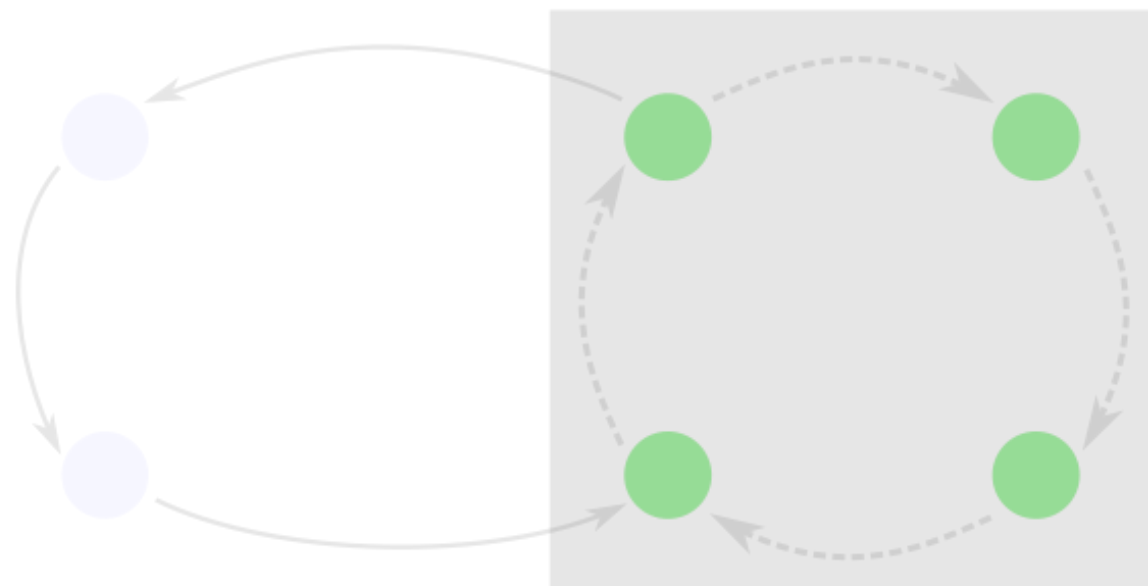


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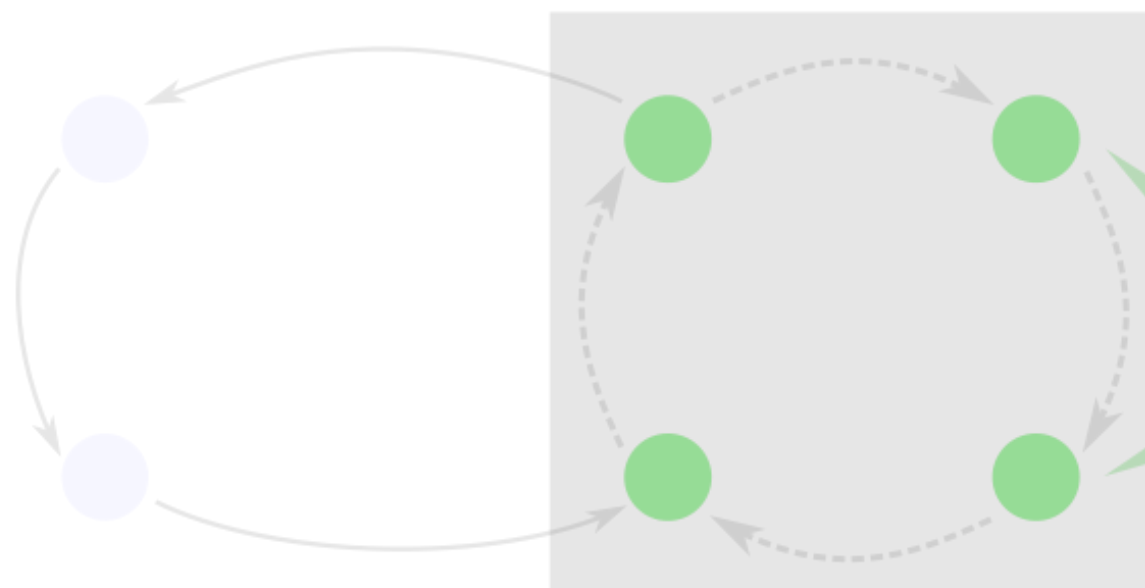


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These agents are strictly better off, and any envied bundles are only shifted around. So, again, EF1 is maintained.

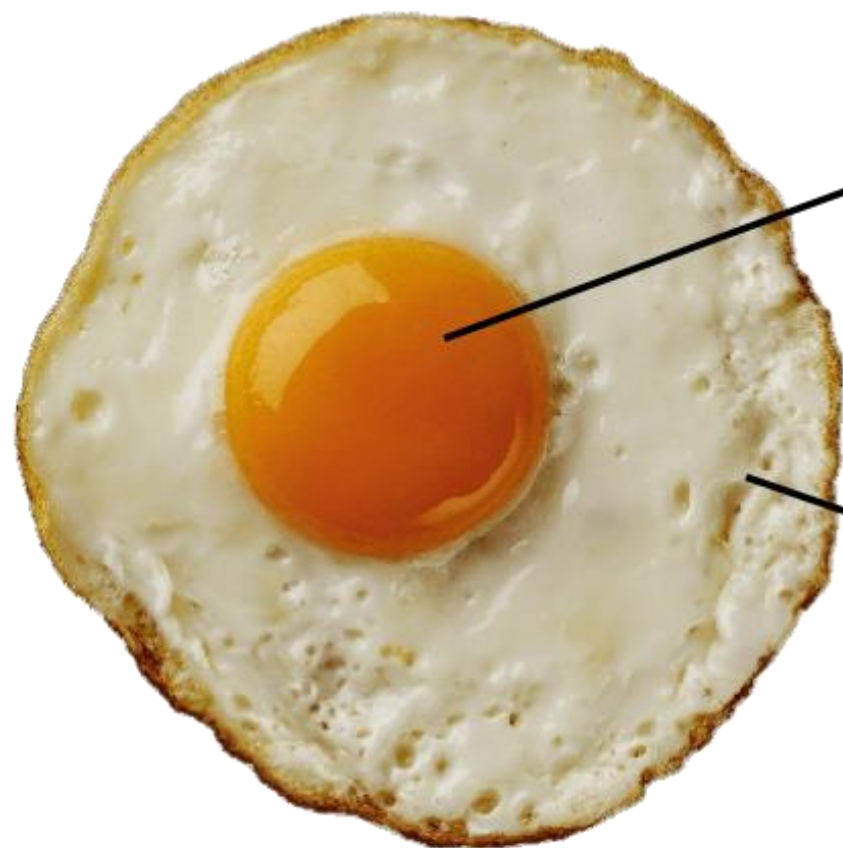
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Additive: $v_i(S) = \sum_{j \in S} v_i(\{j\})$

Monotone: $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$

Next Time

Fairness and Efficiency



References

- Envy-cycle elimination algorithm

Richard Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi
“On Approximately Fair Allocations of Indivisible Goods”

EC 2004, pg 125-131

<https://dl.acm.org/doi/10.1145/988772.988792>

