

Lecture 4

Fairness in Stable Matching Problem

Stable Matching Problem

$w_1 > w_2 > w_3$



$m_3 > m_2 > m_1$

$w_2 > w_1 > w_3$



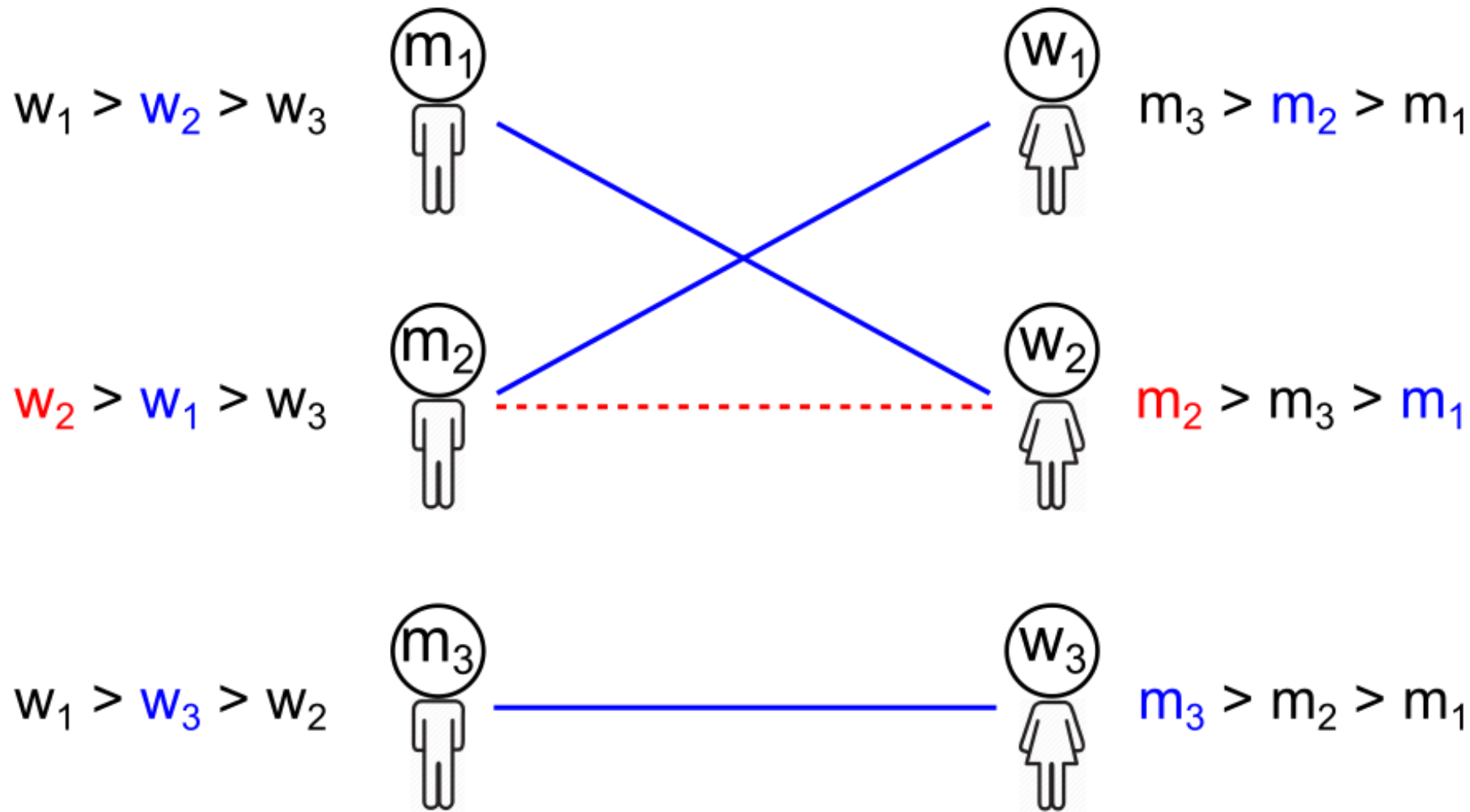
$m_2 > m_3 > m_1$

$w_1 > w_3 > w_2$



$m_3 > m_2 > m_1$

Stable Matching Problem



A matching is **stable** if there is no **blocking pair**.



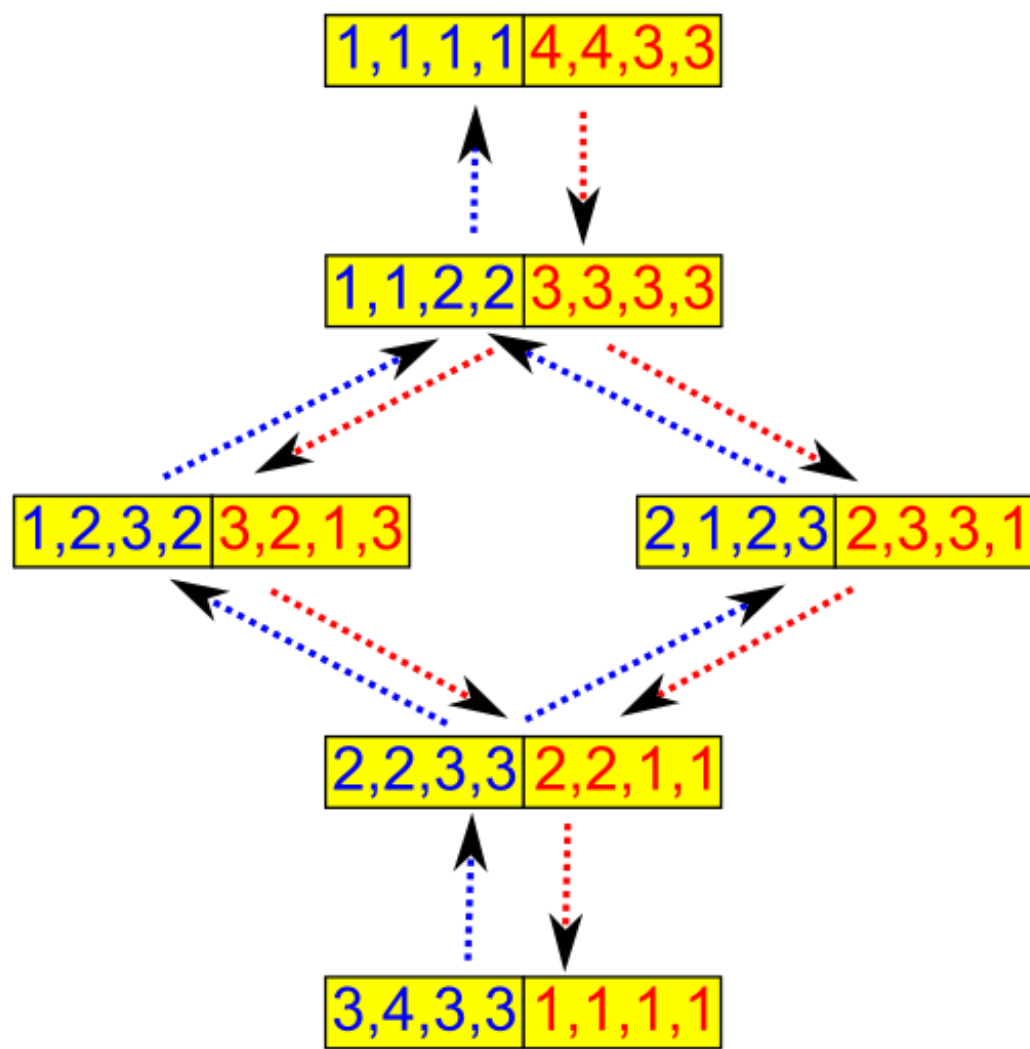
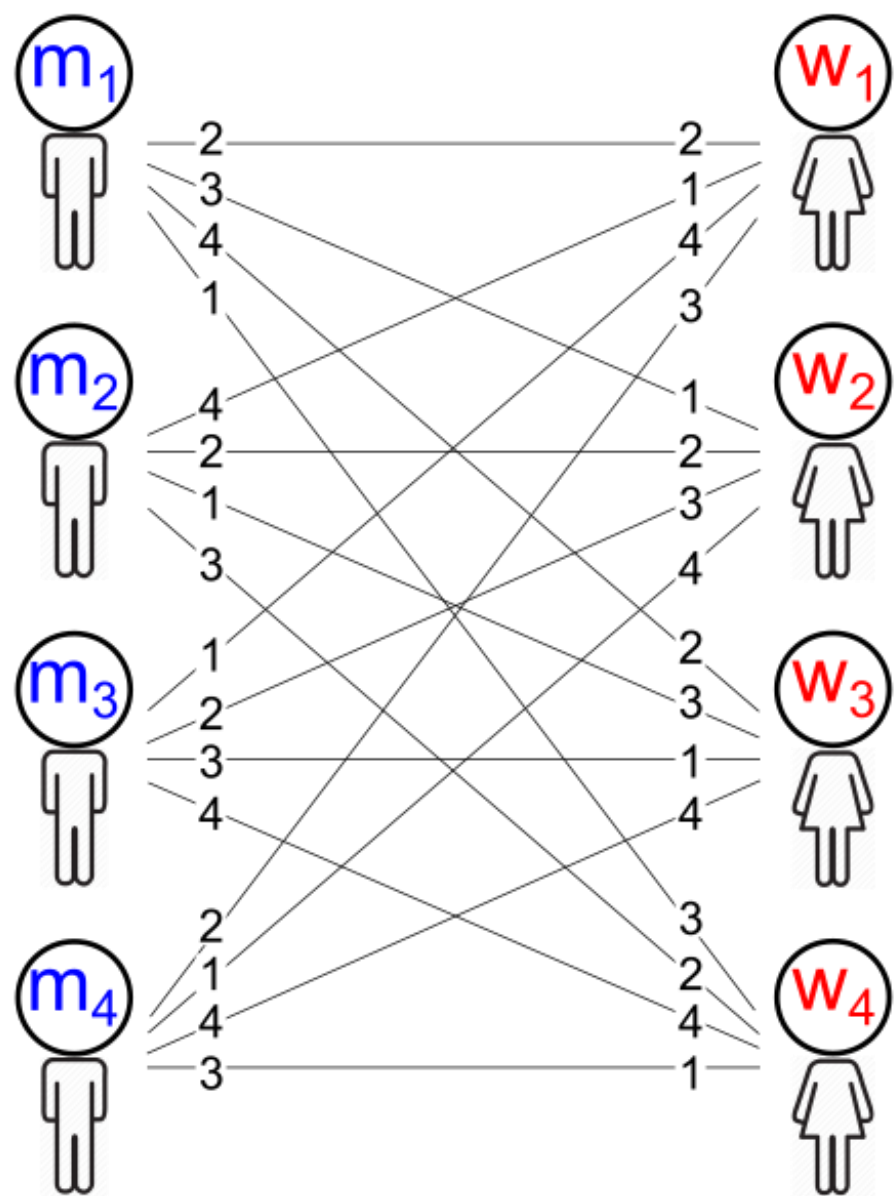
COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

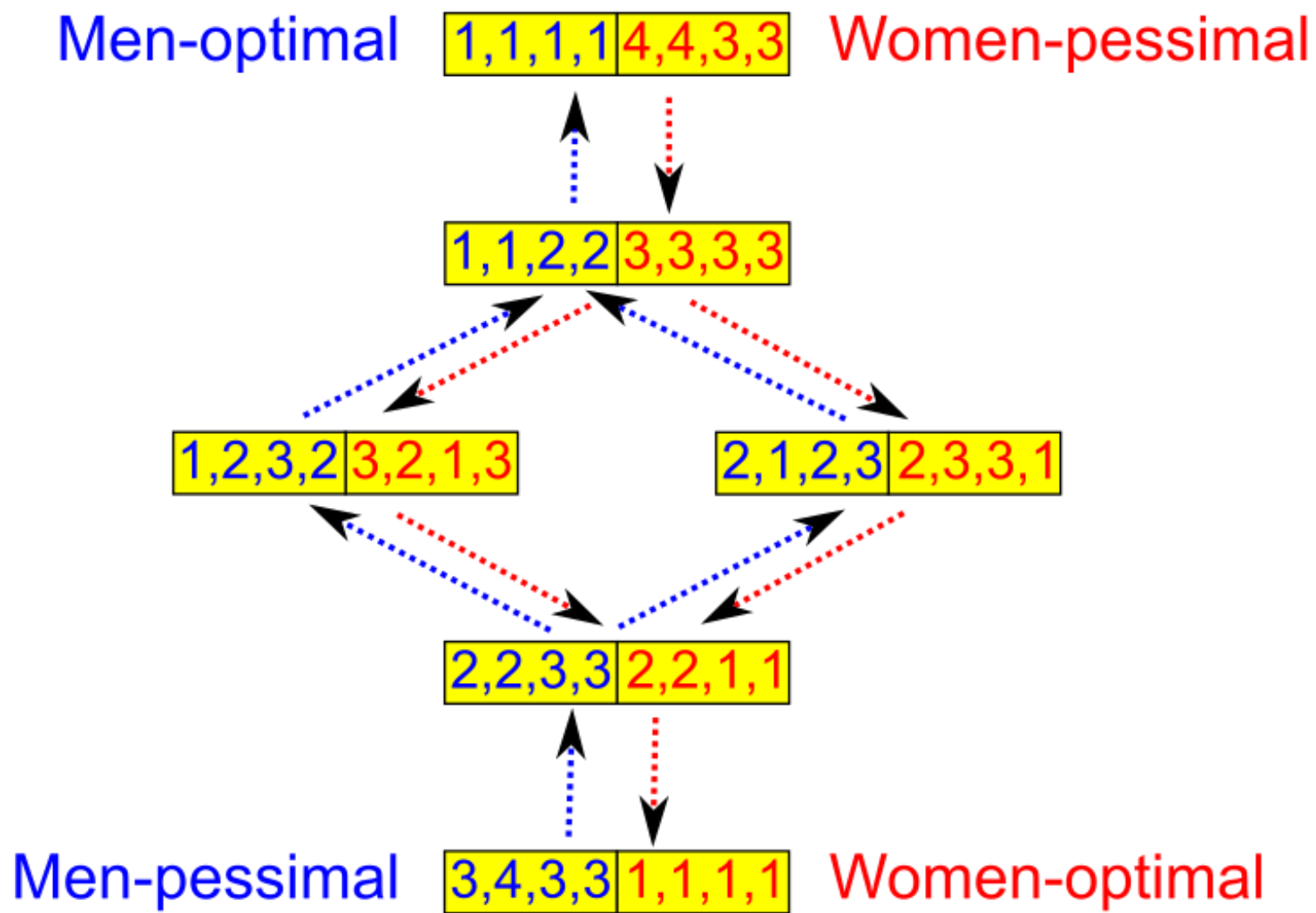
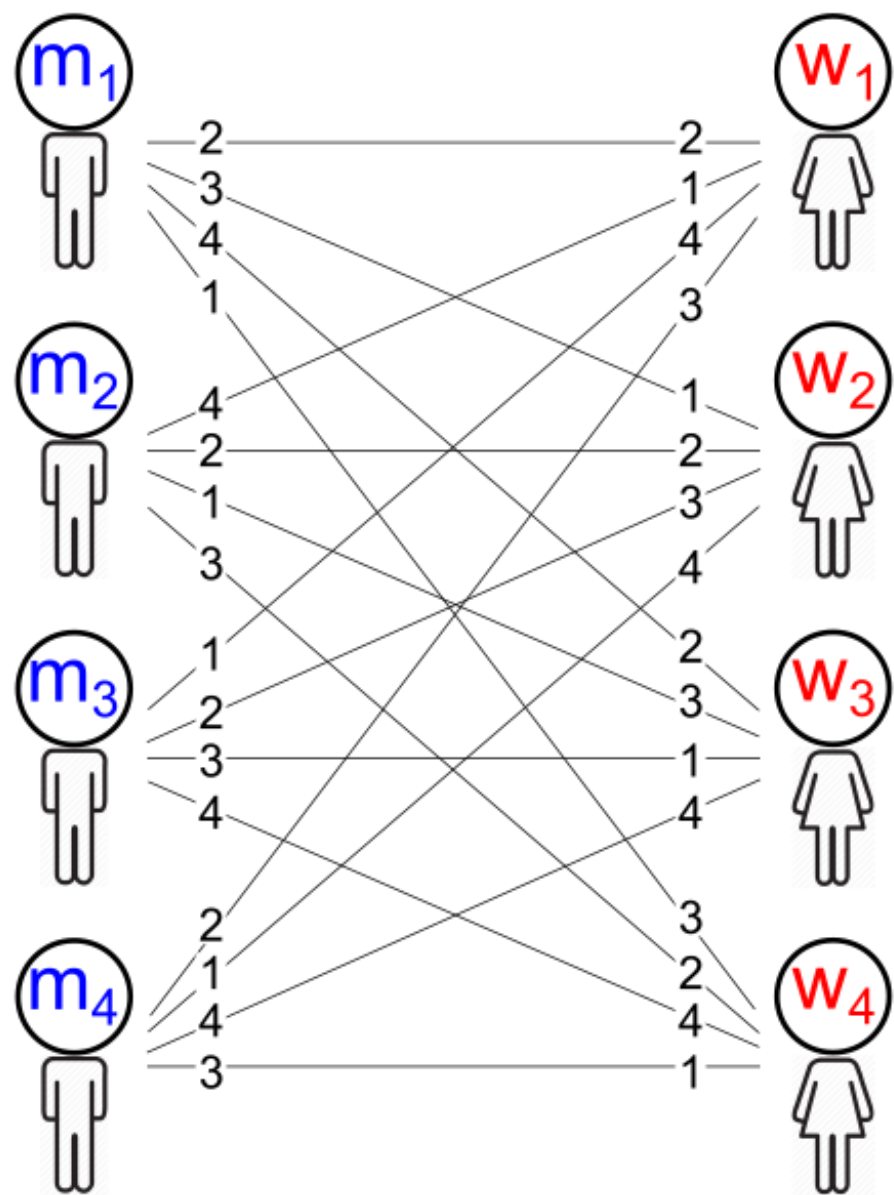
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

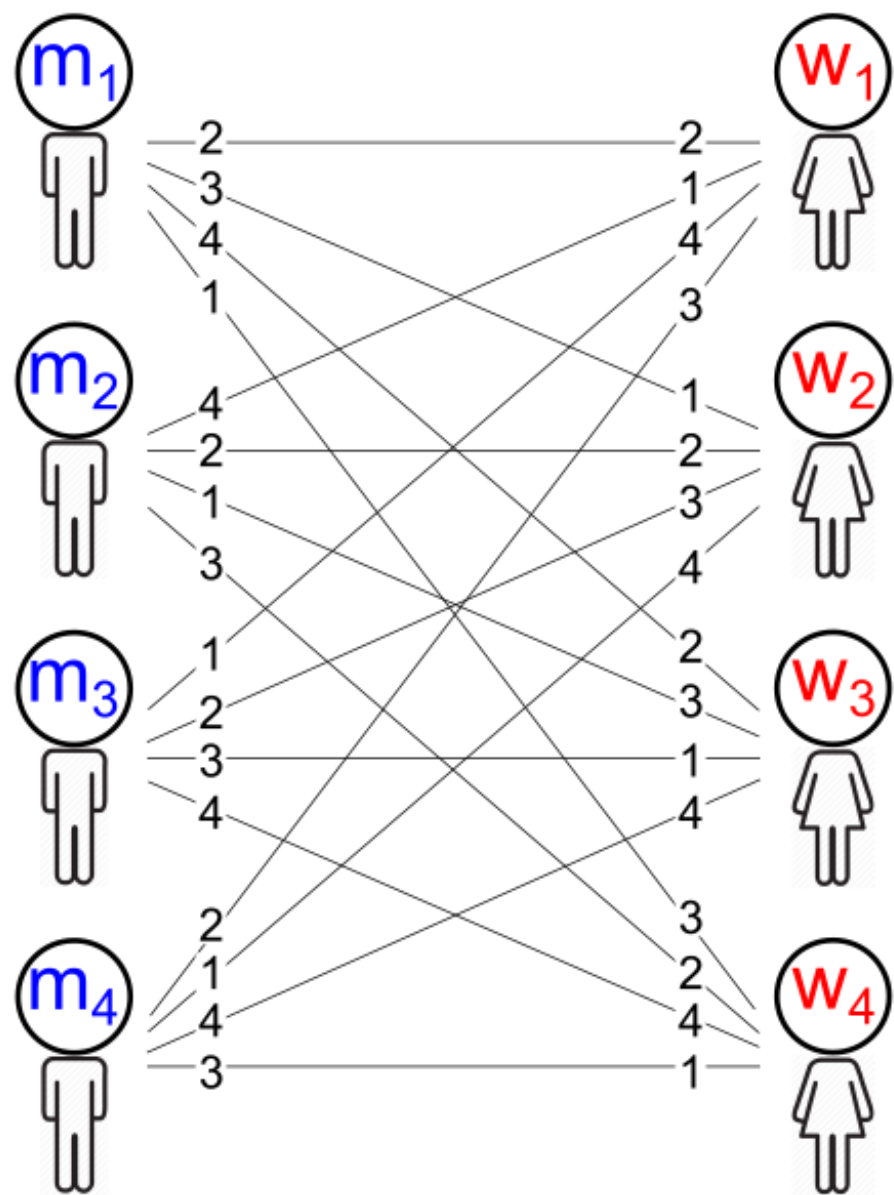


Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9-15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.

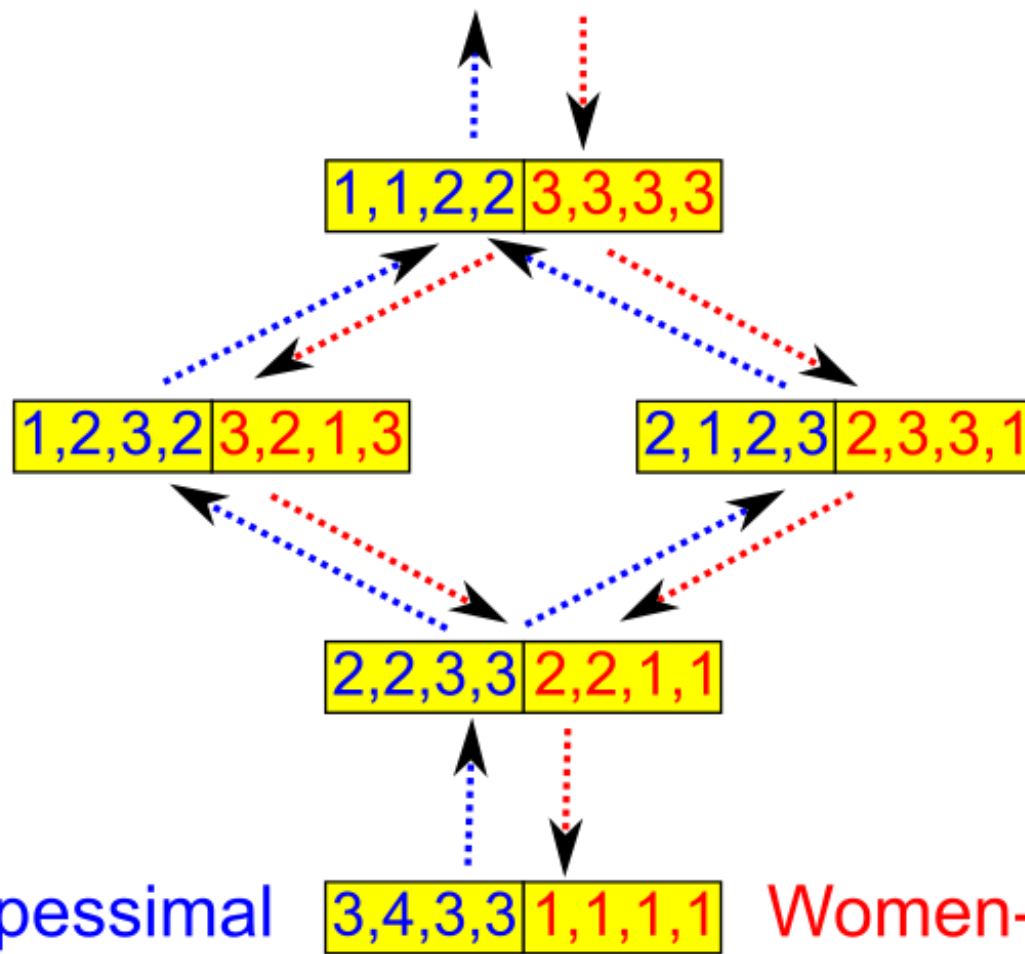






Men-proposing DA algorithm computes this

Men-optimal $1,1,1,1$ $4,4,3,3$ Women-pessimal



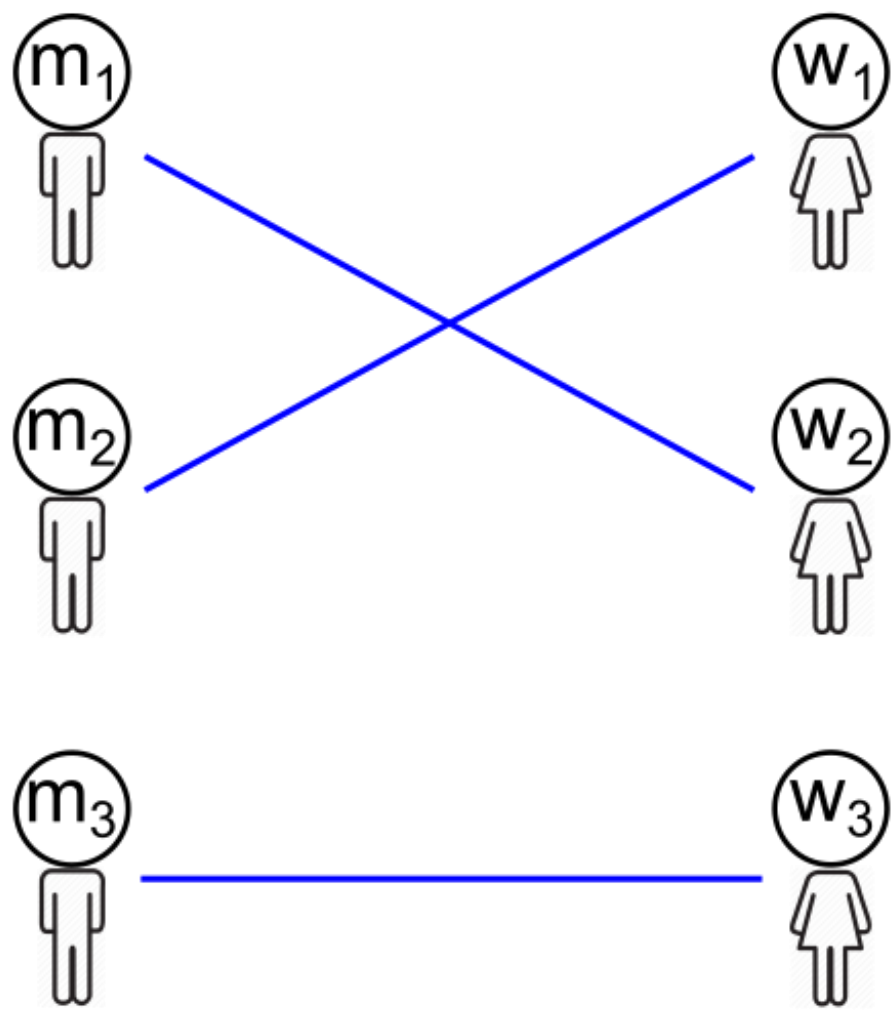
Men-pessimal $3,4,3,3$ $1,1,1,1$ Women-optimal

Women-proposing DA algorithm computes this

Goal for Today

Understanding the structure of the set of stable matchings through linear programming.

(This will guide us towards fair stable matchings.)



$$P = \begin{matrix} & w_1 & w_2 & w_3 \\ m_1 & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ m_2 & \\ m_3 & \end{matrix}$$

Fractional Stable Matching

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Any non-negative $n \times n$ matrix X satisfying the following:

Fractional Stable Matching

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$$X_{i,j} \geq 0 \text{ for all } i \in [n] \text{ and } j \in [n]$$

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$$\sum_j X_{i,j} = 1 \text{ for all } i \in [n] \quad \textit{Every man is fully matched.}$$

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$$\sum_i X_{i,j} = 1 \text{ for all } j \in [n] \quad \textit{Every woman is fully matched.}$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \text{ for all } i \in [n] \text{ and } j \in [n] \quad \textit{Stability}$$

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Any integral stable matching is also a fractional stable matching.

Fractional Stable Matching

Any non-negative $n \times n$ matrix X satisfying the following:

Any convex combination of integral stable matchings is also a fractional stable matching.

$$X = \sum_k \lambda_k P^k$$

integral stable matching

such that $\lambda_k \geq 0$ for all k and $\sum_k \lambda_k = 1$

Any integral stable matching is also a fractional stable matching.

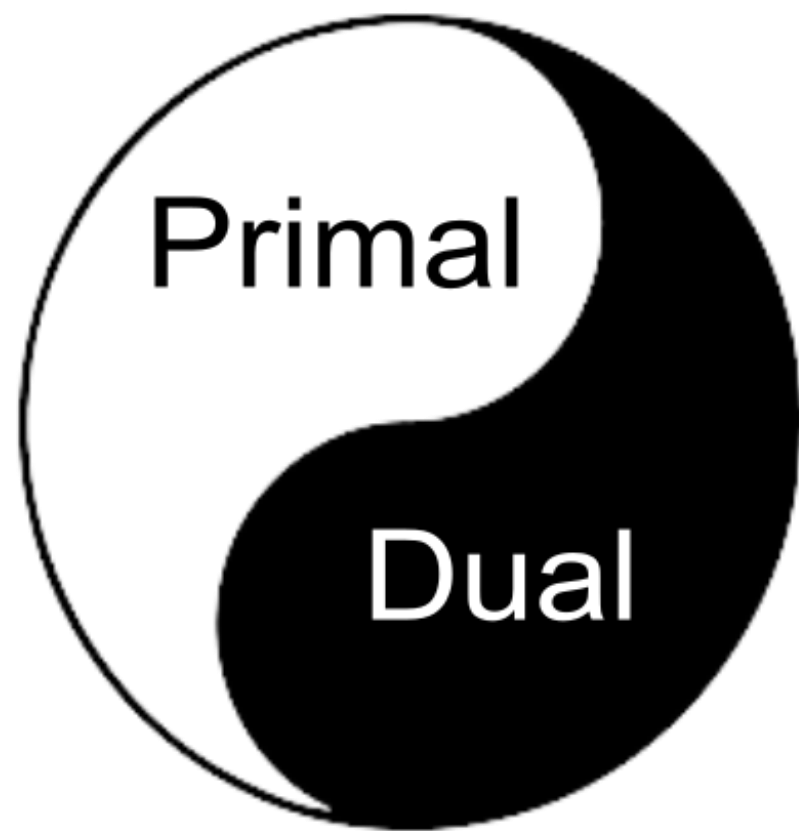
[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Coming up...

An elegant geometric proof that uses LP duality and its application in fair stable matchings.



Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Primal

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$$X_{i,j} \geq 0 \quad \forall i, j$$

Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Primal

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Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i \quad \alpha_i$$

$$\sum_i X_{i,j} = 1 \quad \forall j \quad \beta_j$$

$\gamma_{i,j}$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

Don't worry about us.

Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i \quad \alpha_i$$

Combine the constraints in order to construct an upper bound on the objective.

$$-X_{i,j} - \sum_{k:w_k \succ m_i} X_{i,k} - \sum_{k:m_k \succ w_j} X_{k,j} \leq -1 \quad \forall i,j \quad \gamma_{i,j}$$

$$X_{i,j} \geq 0 \quad \forall i,j$$

Don't worry about us.

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Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$\gamma_{i,j}$

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$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k:w_k > m_i w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k > w_j m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

Primal

Let's combine these.

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

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$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k:w_k > m_i w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k > w_j m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

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 $\gamma_{i,j} \geq 0 \quad \forall i, j$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

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$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j > m_i w_k} \gamma_{i,k} - \sum_{k: m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

subject to this
being at least 1

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

as long as
 $\gamma_{i,j} \geq 0 \forall i, j$

subject to this
being at least 1

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

By Gale and Shapley's result,
primal is always feasible!

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

Let $X_{i,j}$ be a feasible primal solution.

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\boxed{\sum_j X_{i,j}} + \sum_i X_{i,j} - \boxed{X_{i,j}} - \boxed{\sum_{k:w_j \succ_{m_i} w_k} X_{i,k}} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

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$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_i X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

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$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$$

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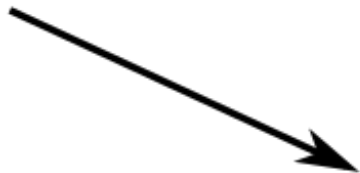
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Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint
from primal



$$\sum_{i,j} X_{i,j}$$

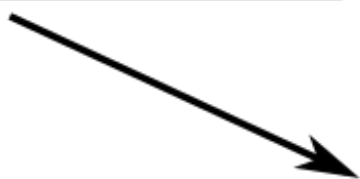
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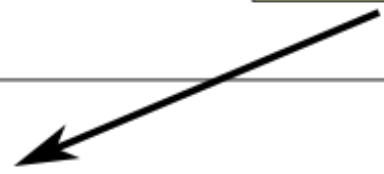


$$\sum_{i,j} X_{i,j}$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

Dual feasible!



Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

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Stability constraint
from primal

Objective is equal
to that in the primal

$$\sum_{i,j} X_{i,j}$$

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Dual

Stability constraint
from primal

Objective is equal
to that in the primal

$$\sum_{i,j} X_{i,j}$$

By strong duality:

$X_{i,j}$ must be *primal optimal*, and $X_{k,j} + X_{i,j} \geq 1 \quad \forall i, j$

$\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ must be *dual optimal*.

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint
from primal

Objective is equal
to that in the primal

By complementary slackness:

For any primal feasible X ,

$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k: w_k > m_i w_j} X_{i,k} + \sum_{k: m_k > w_j m_i} X_{k,j} = 1.$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

[Vande Vate, *Oper. Res. Let.* 1989]

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Proof by picture (and LP duality)

[Teo and Sethuraman, *MOR* 1998].

[Vande Vate, *Oper. Res. Let.* 1989]

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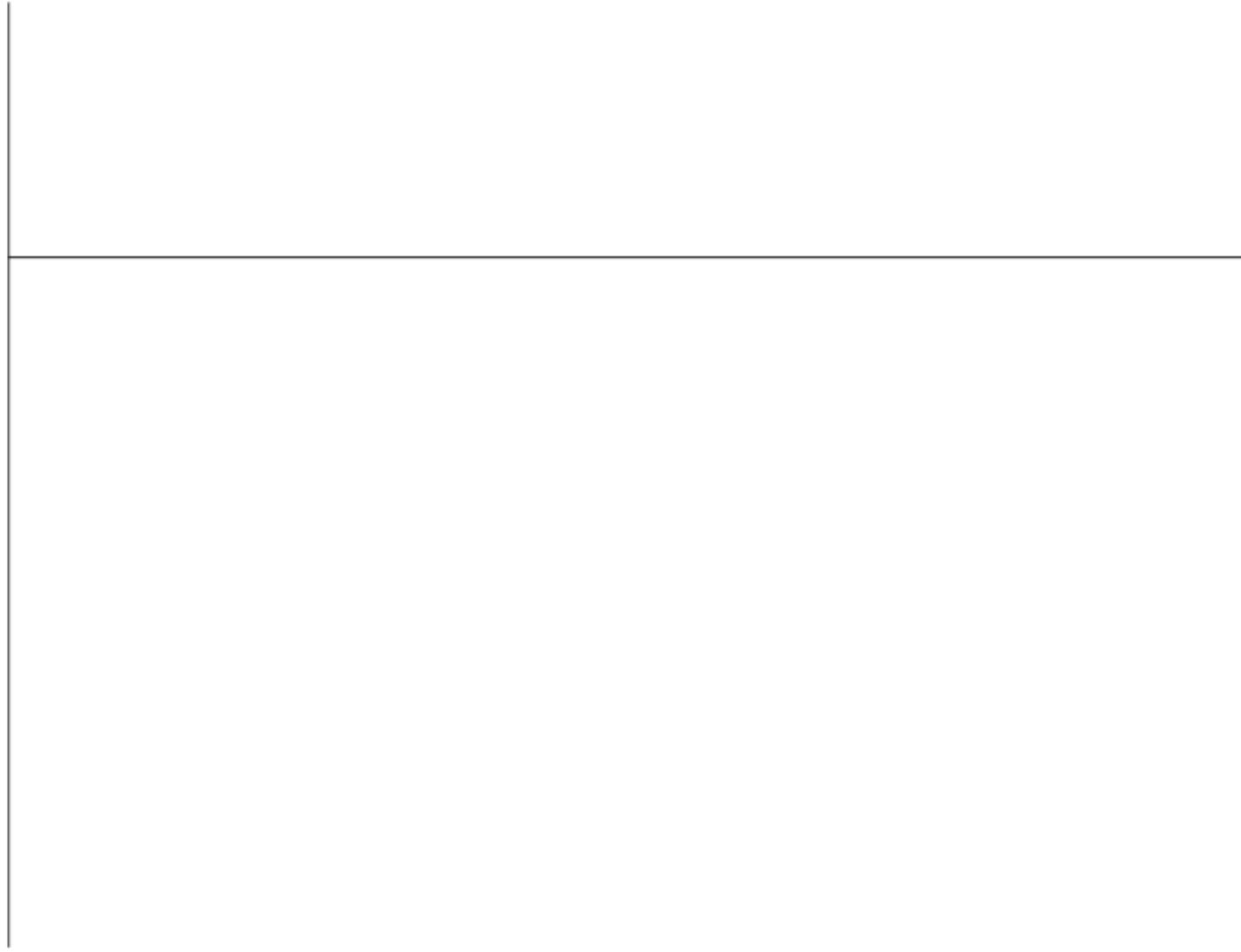
Recall complementary slackness:

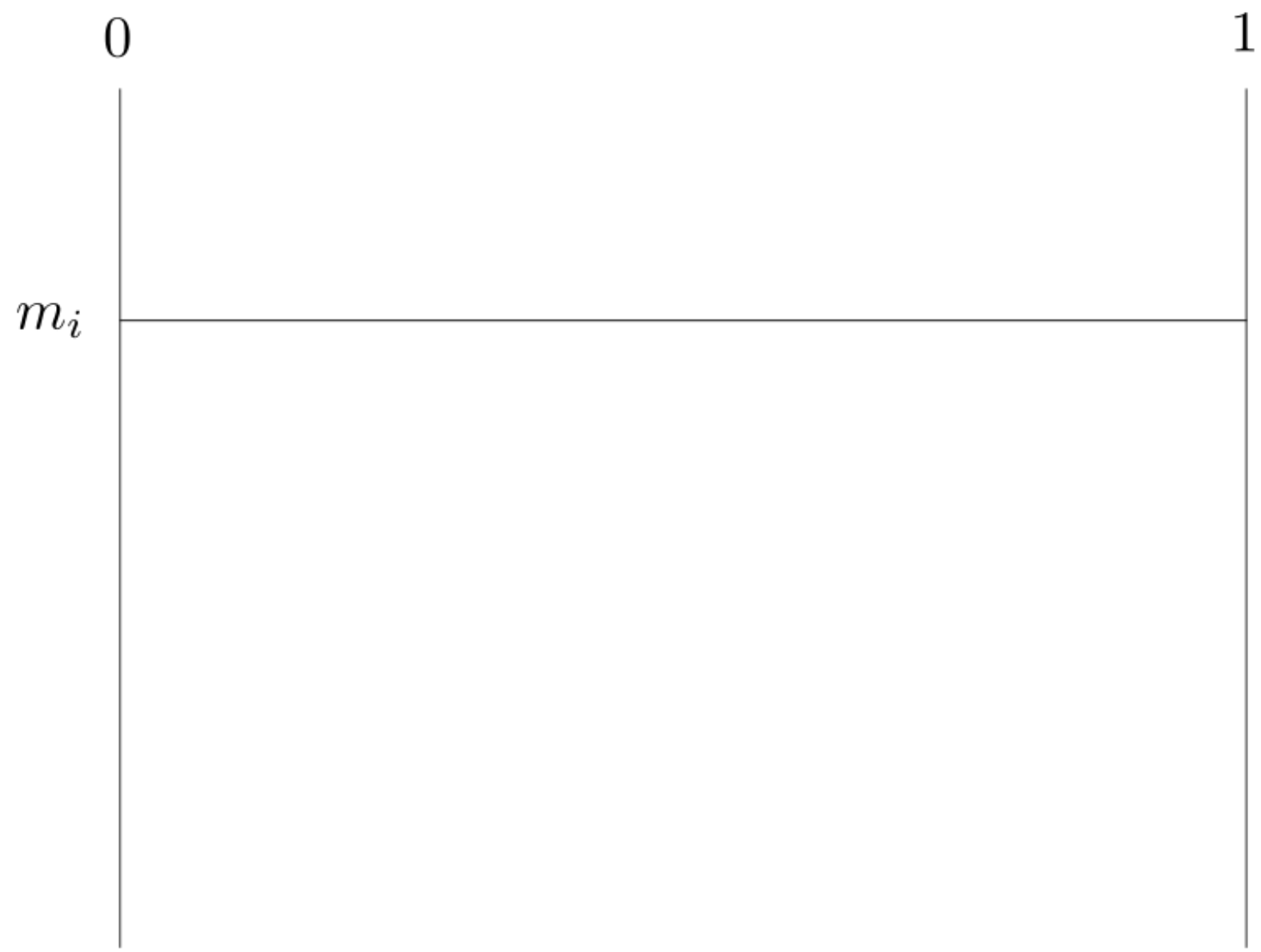
For any primal feasible X ,

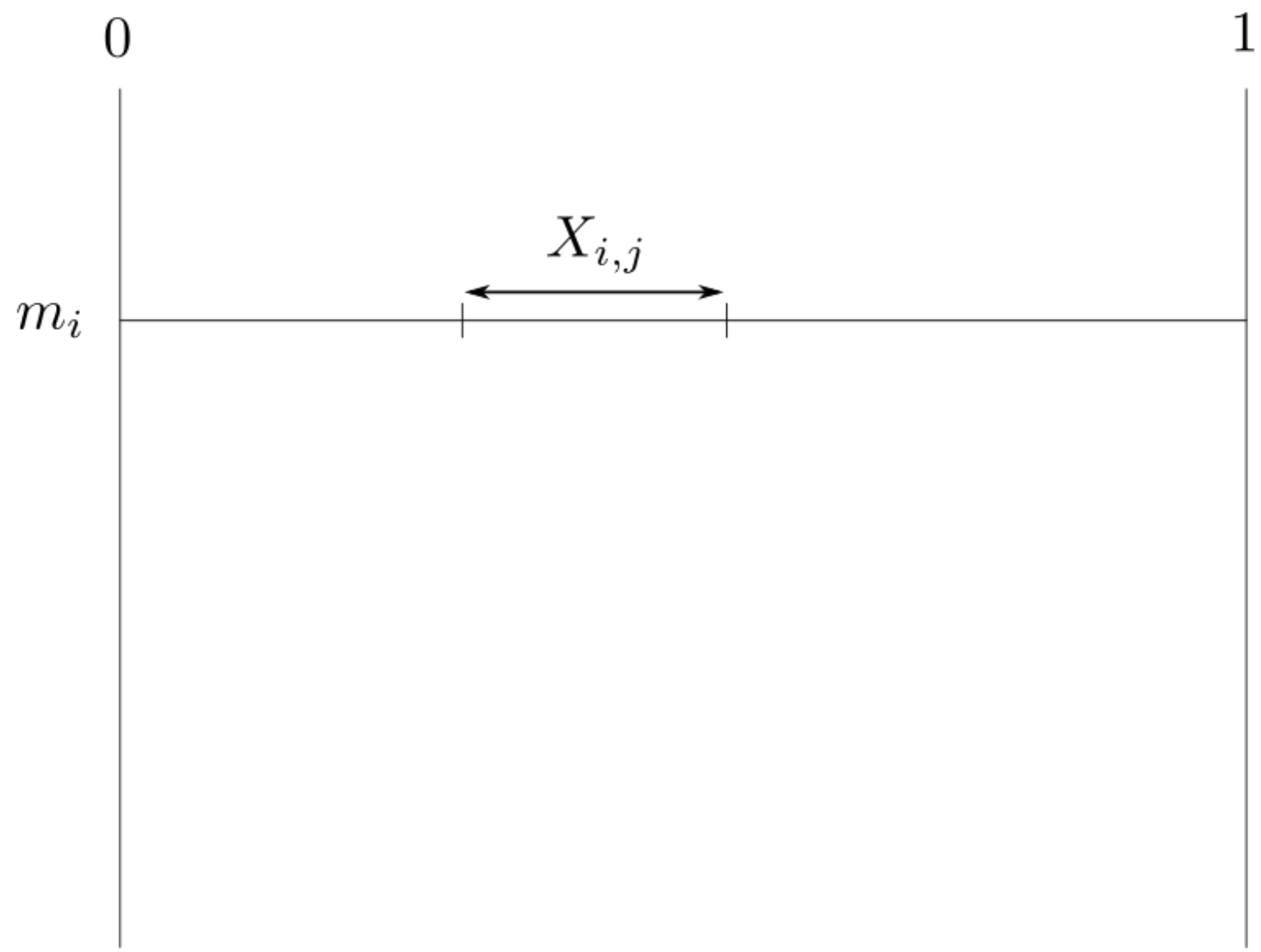
$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} = 1.$$

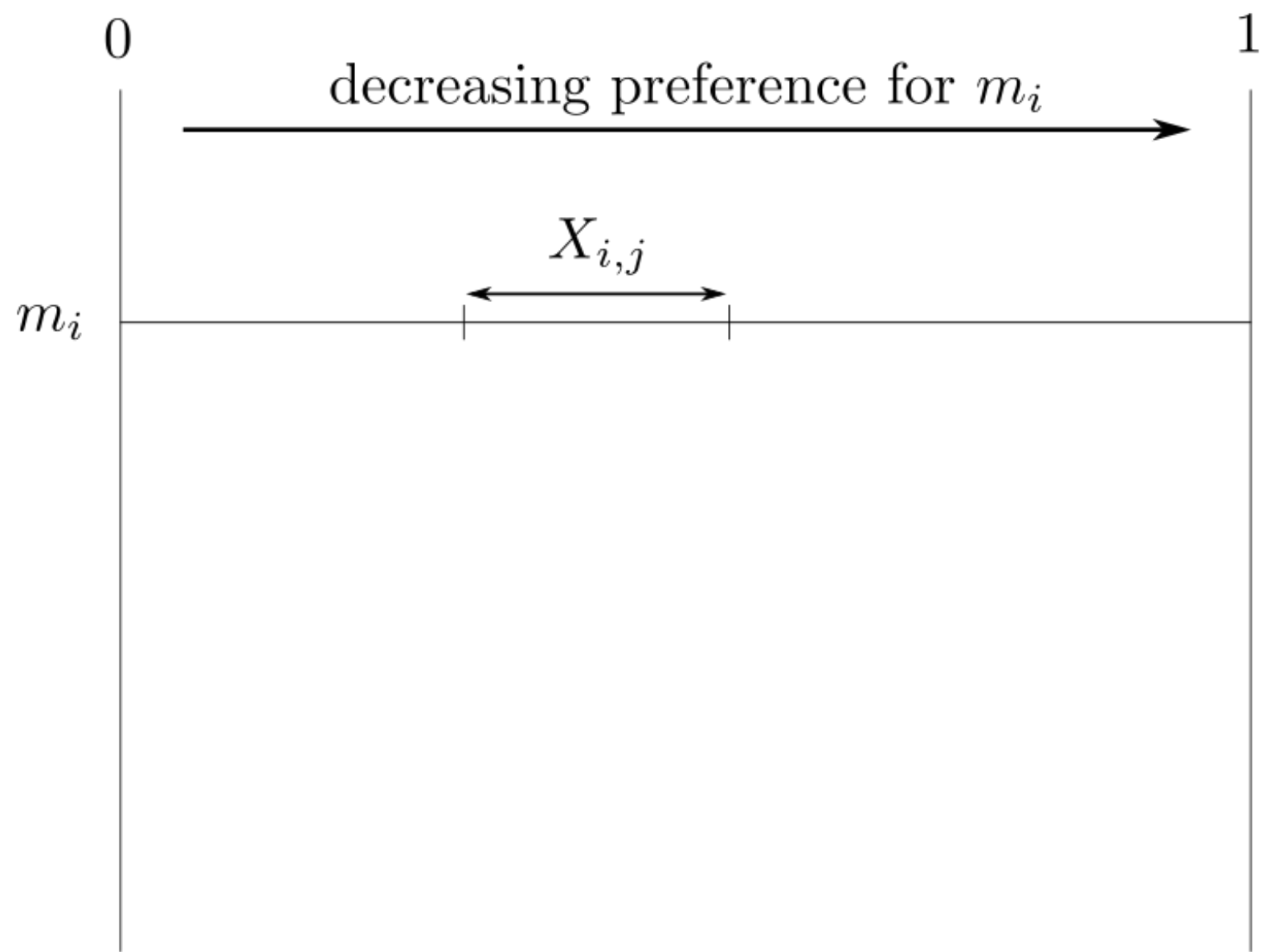
0

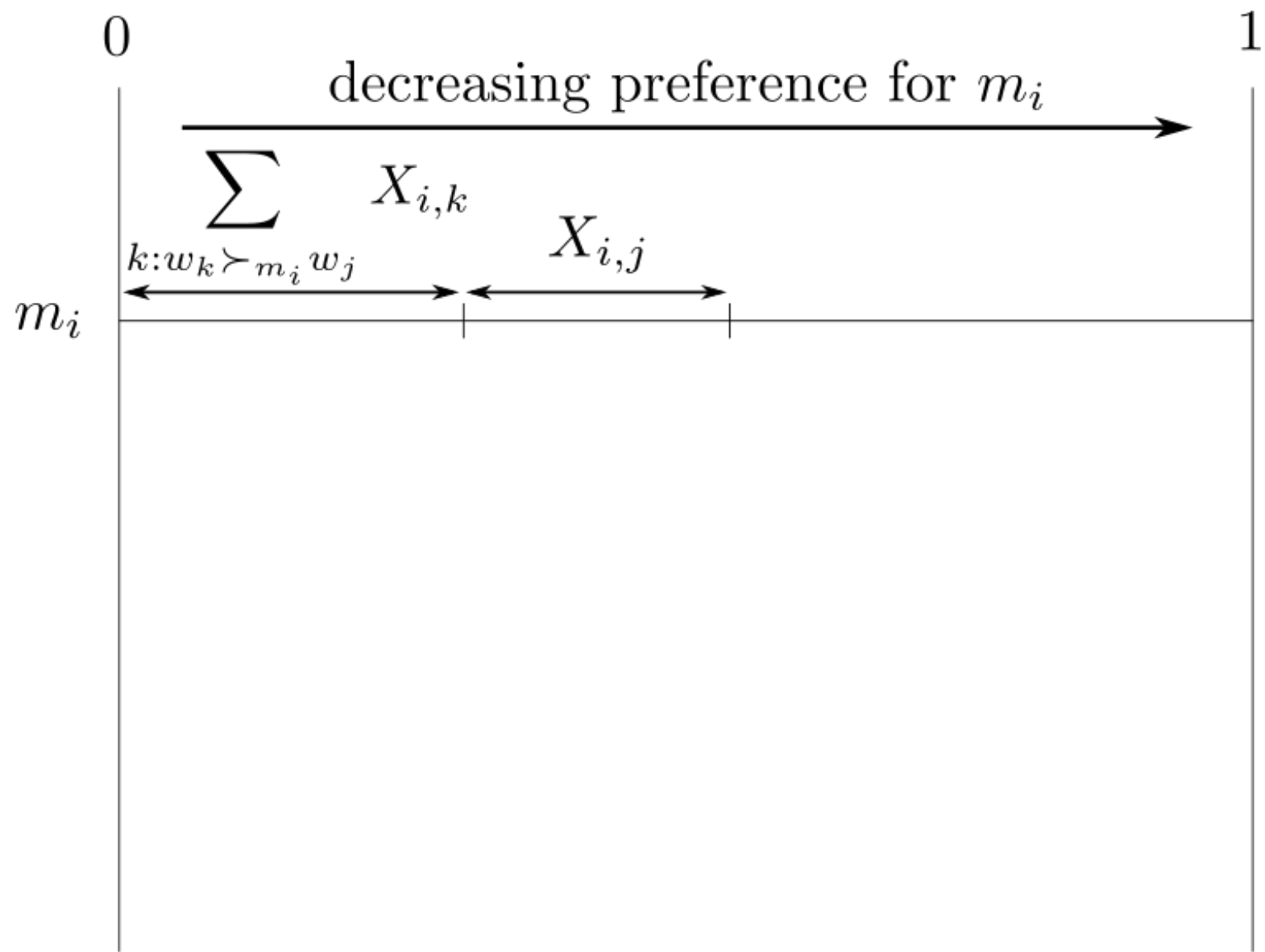
1

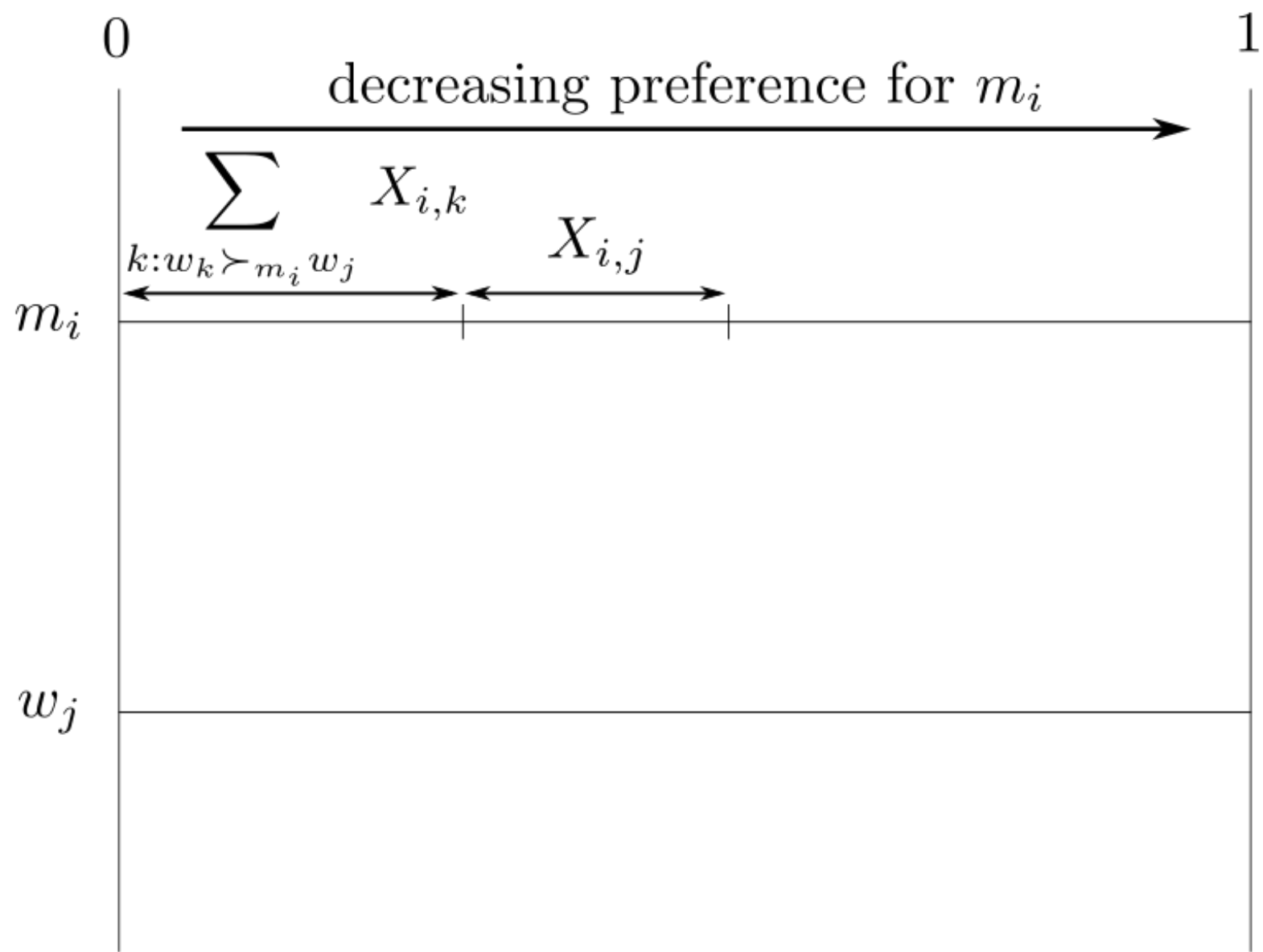


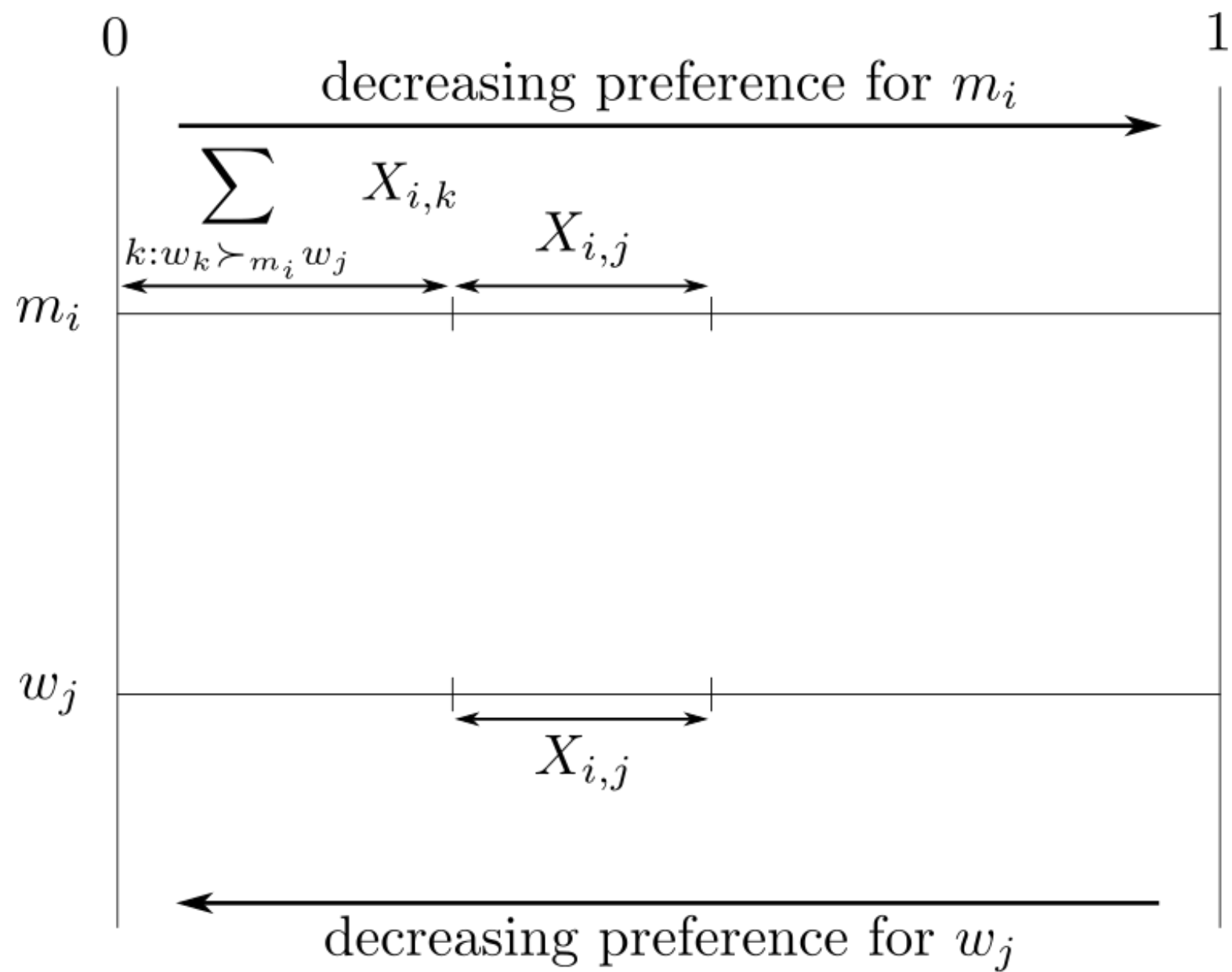


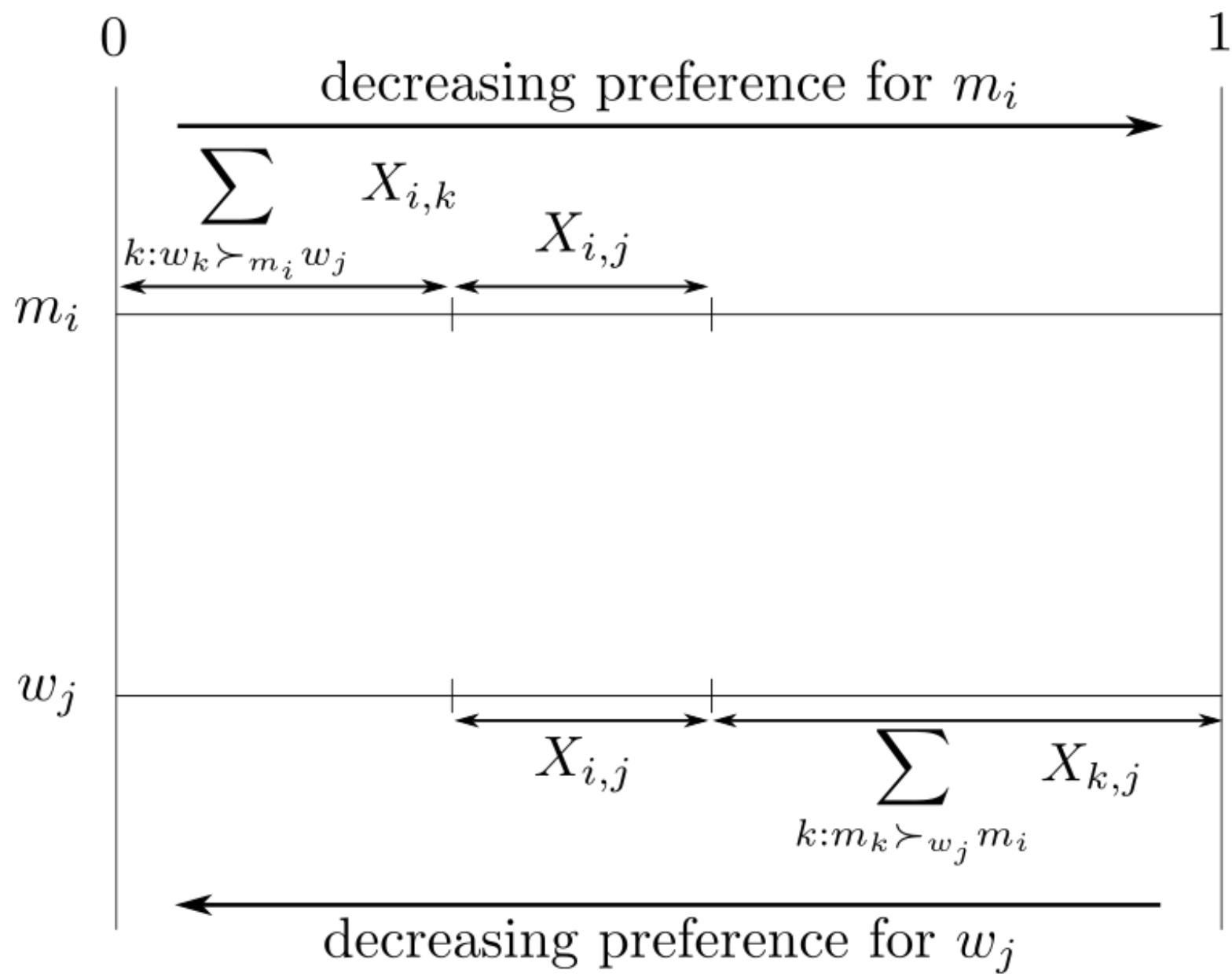


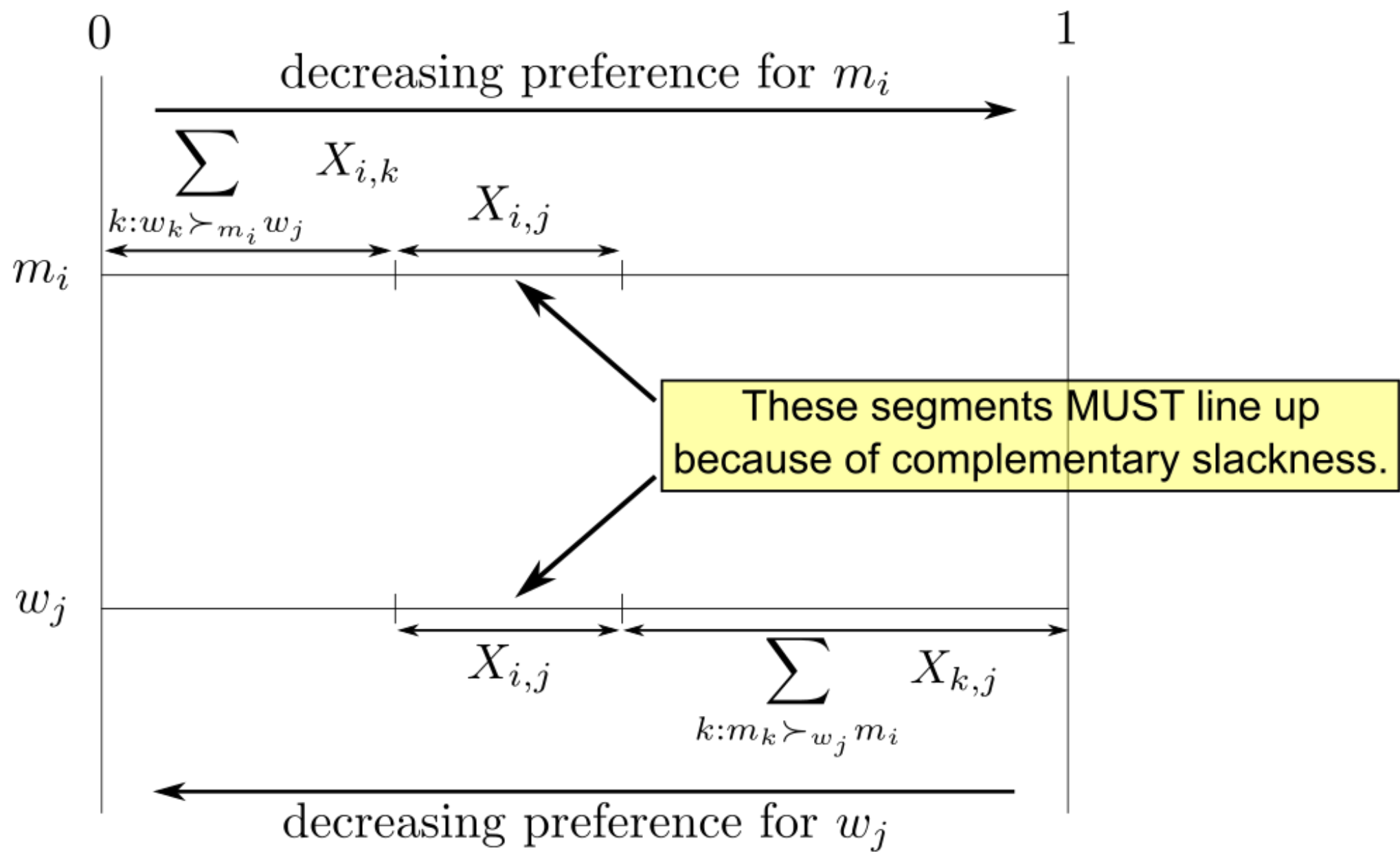


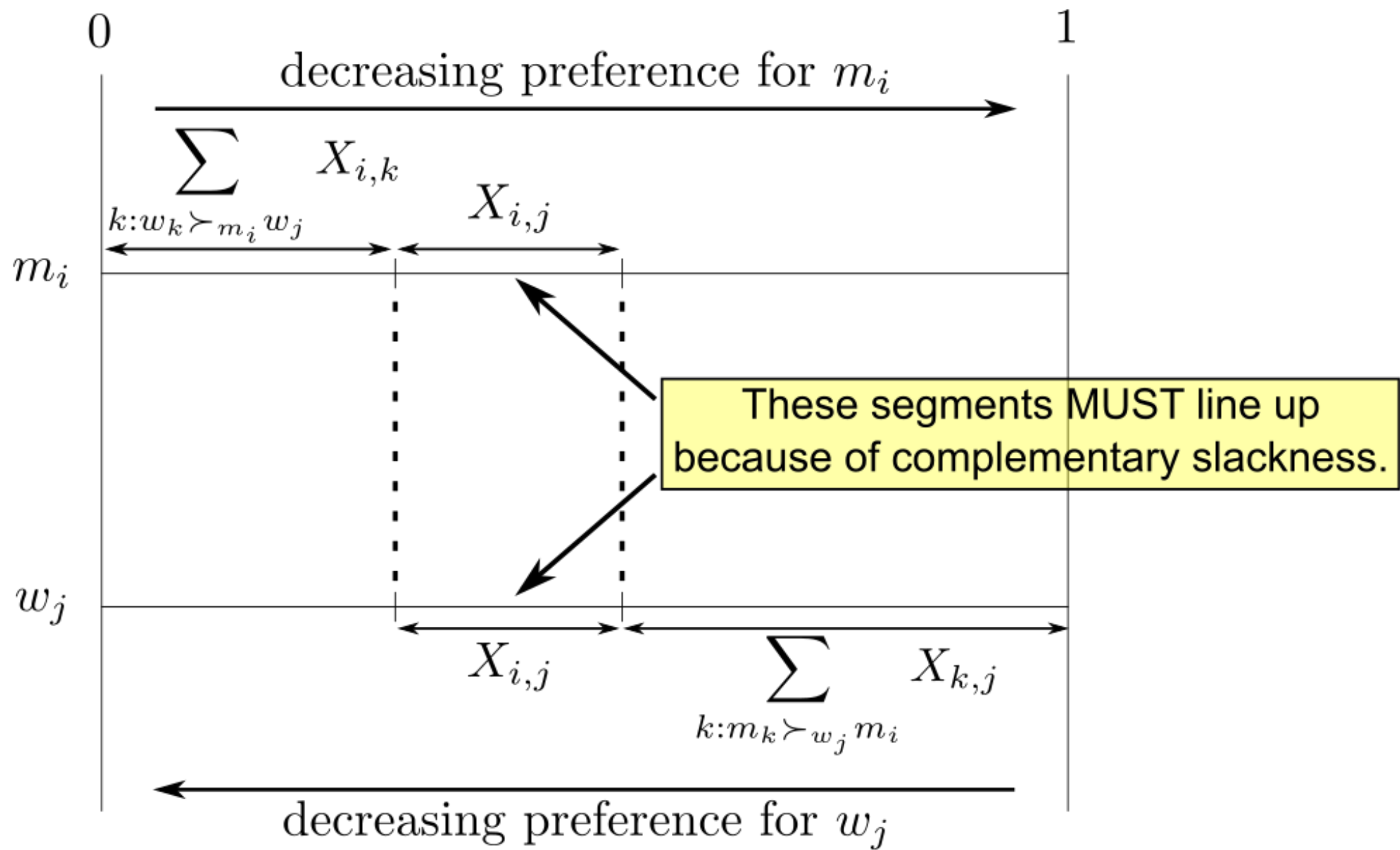


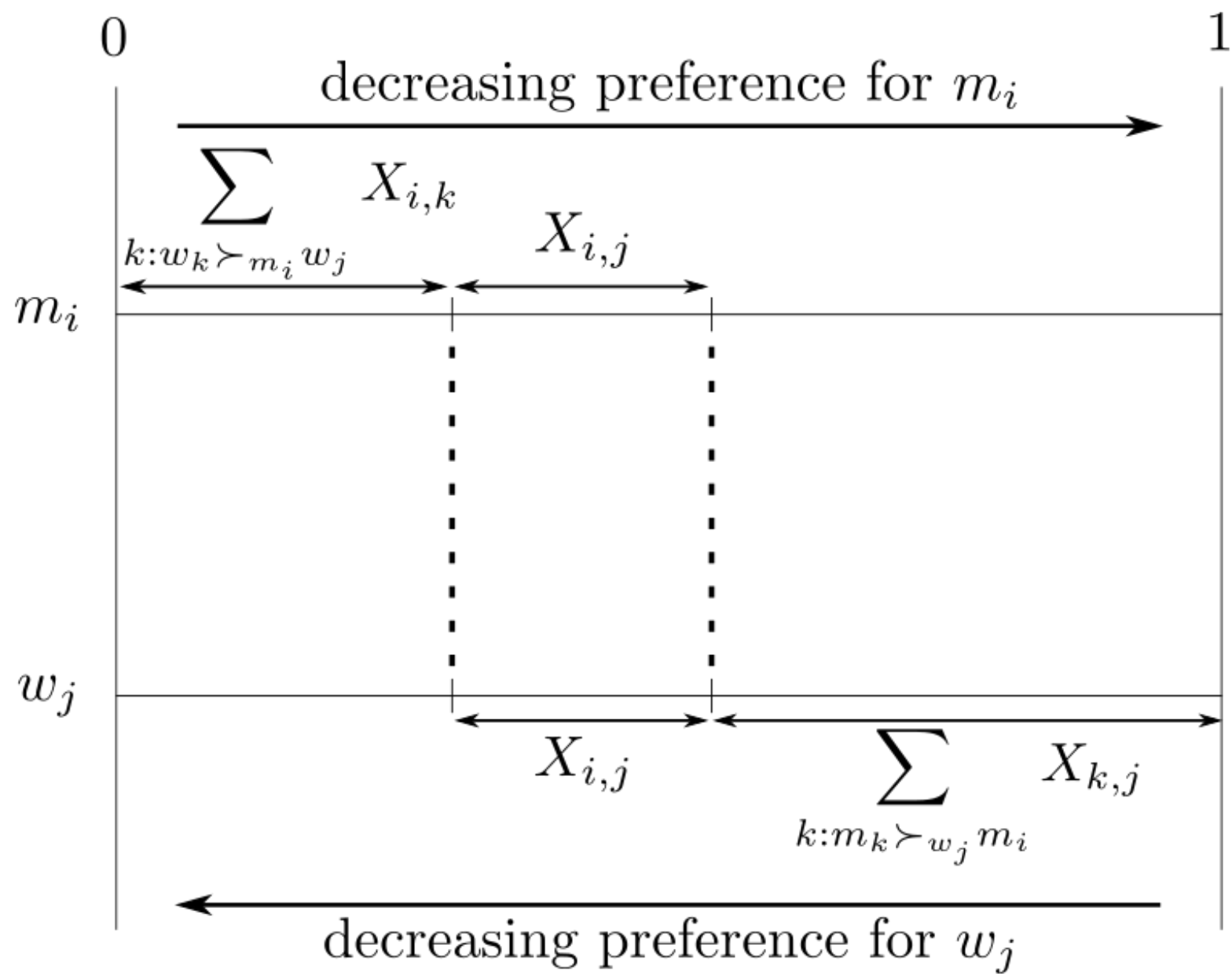


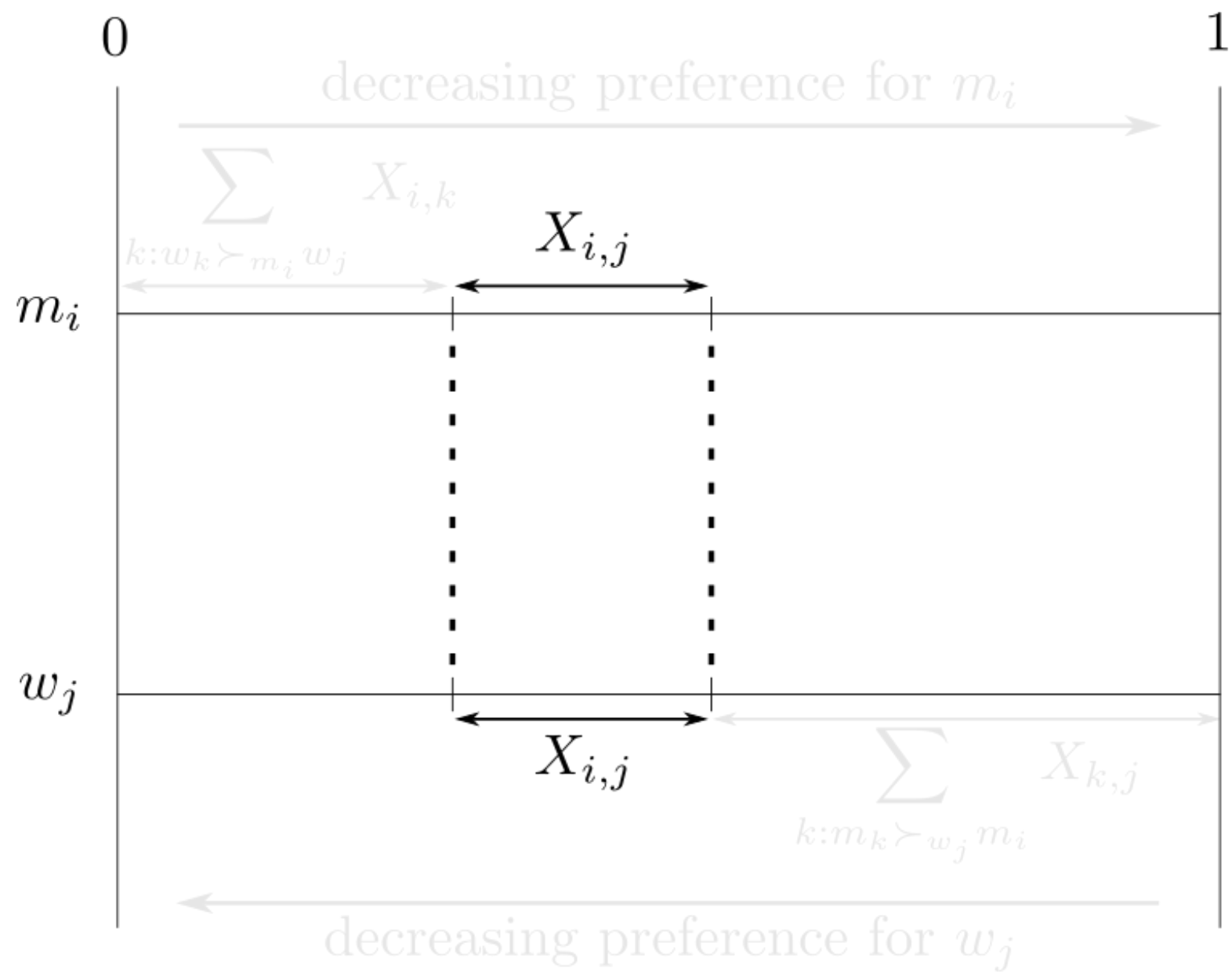


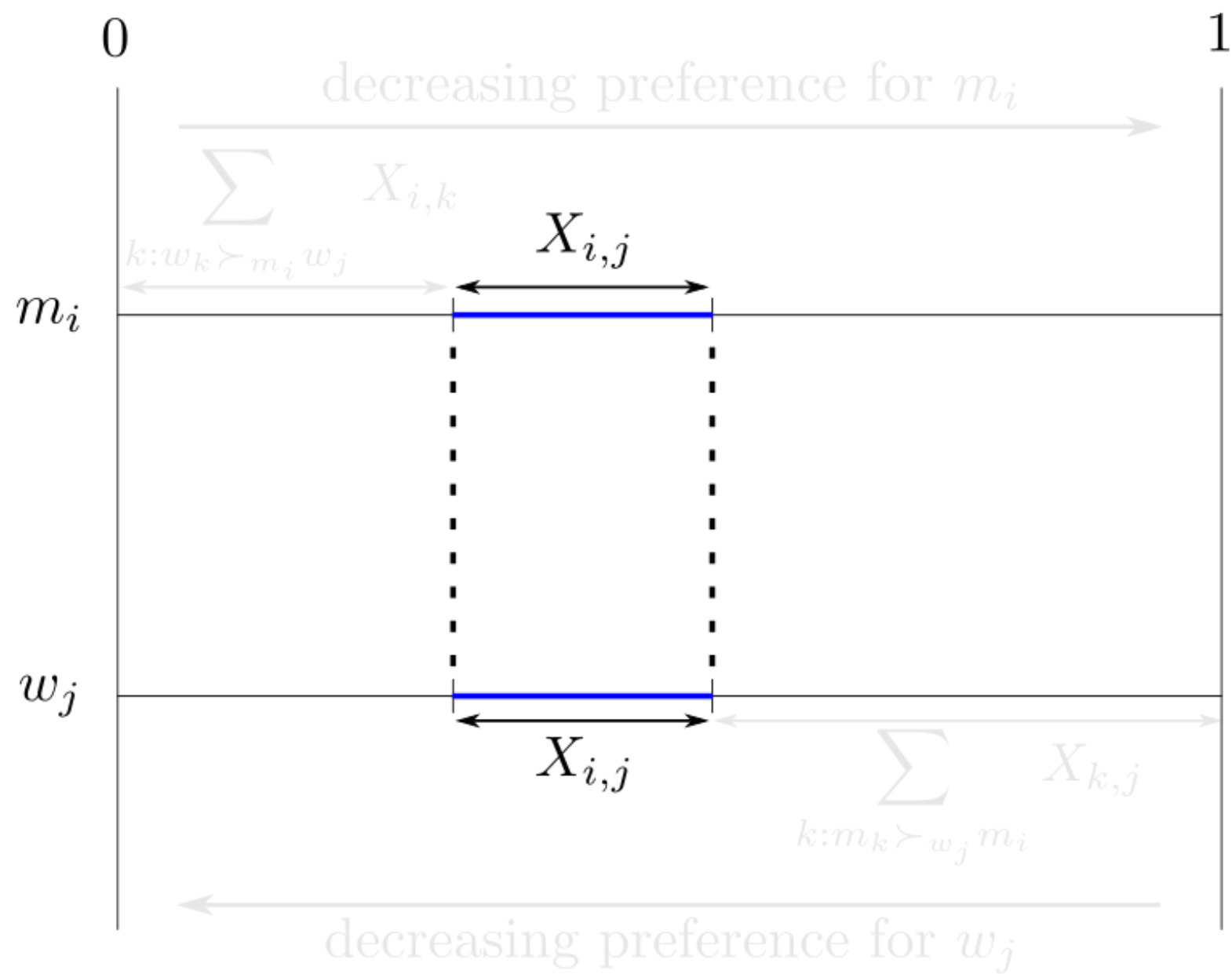


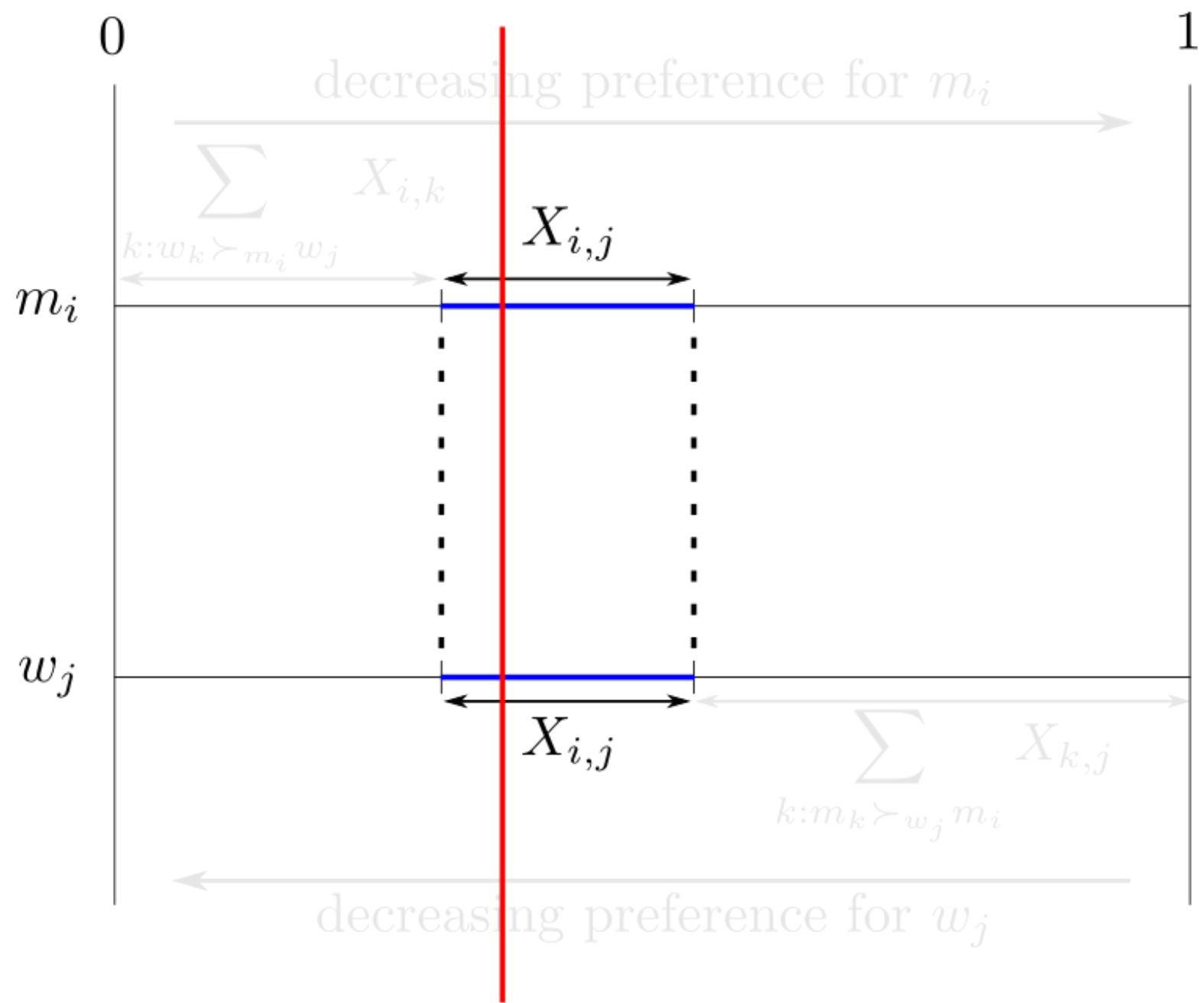


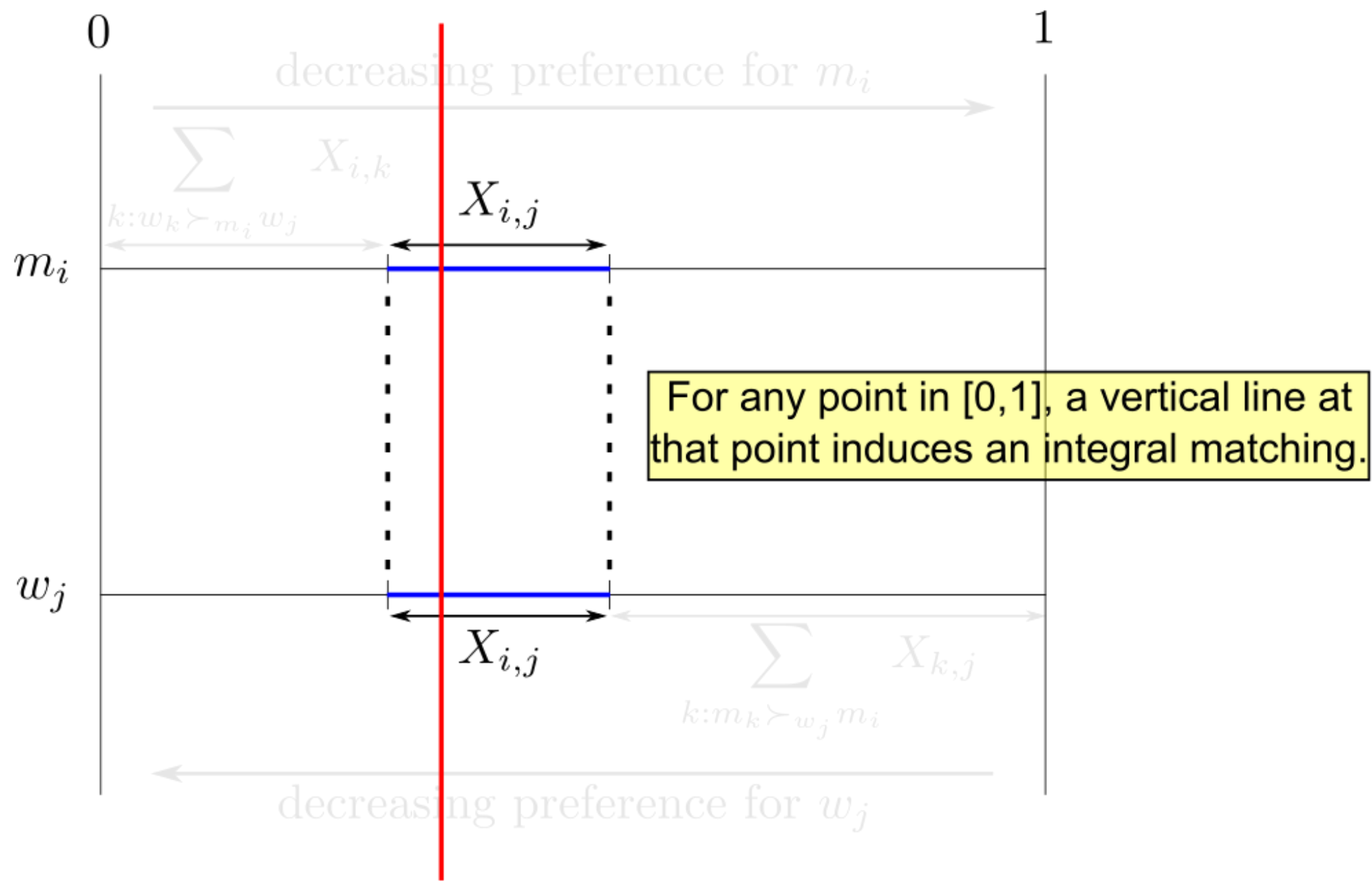


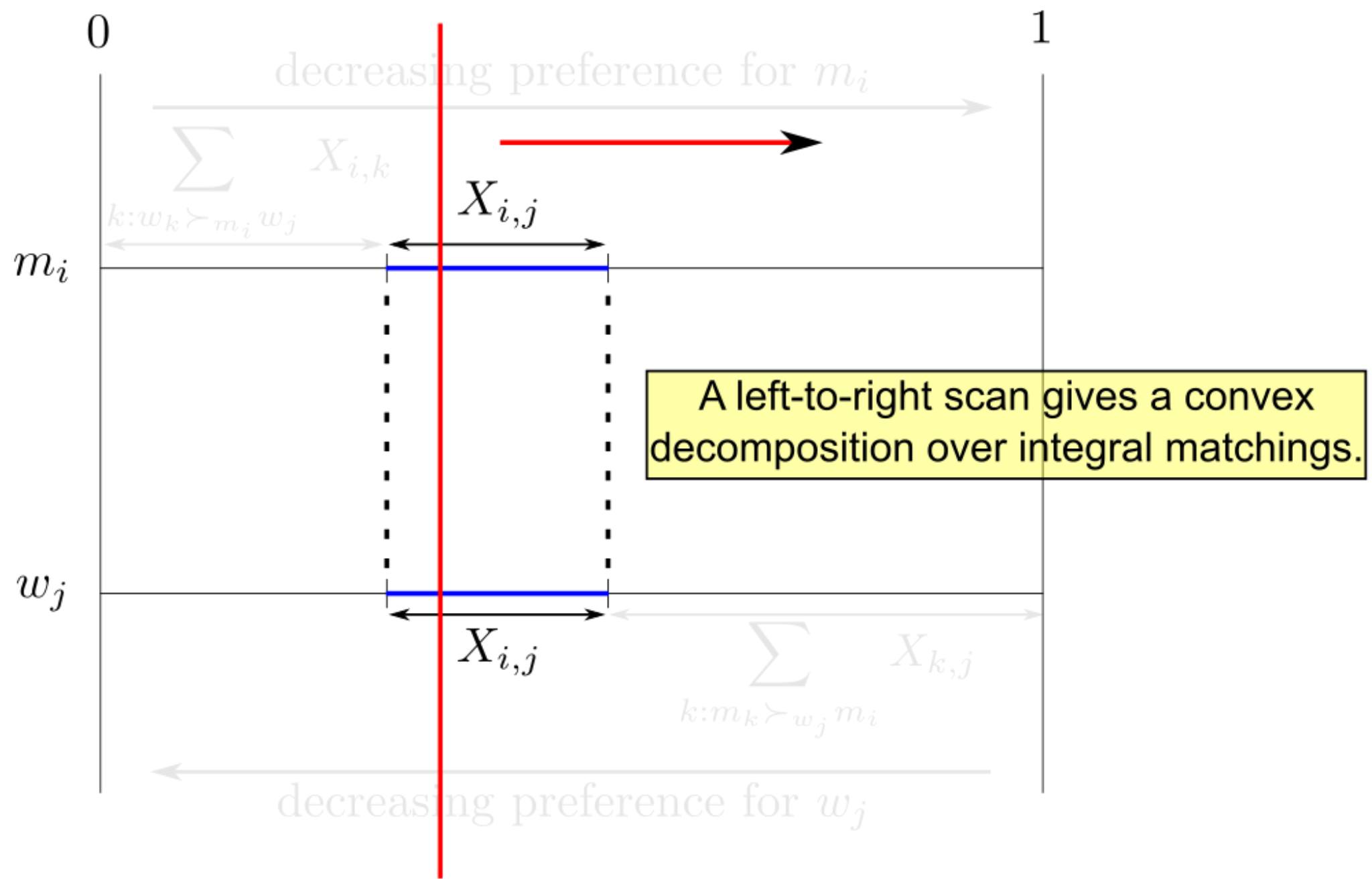


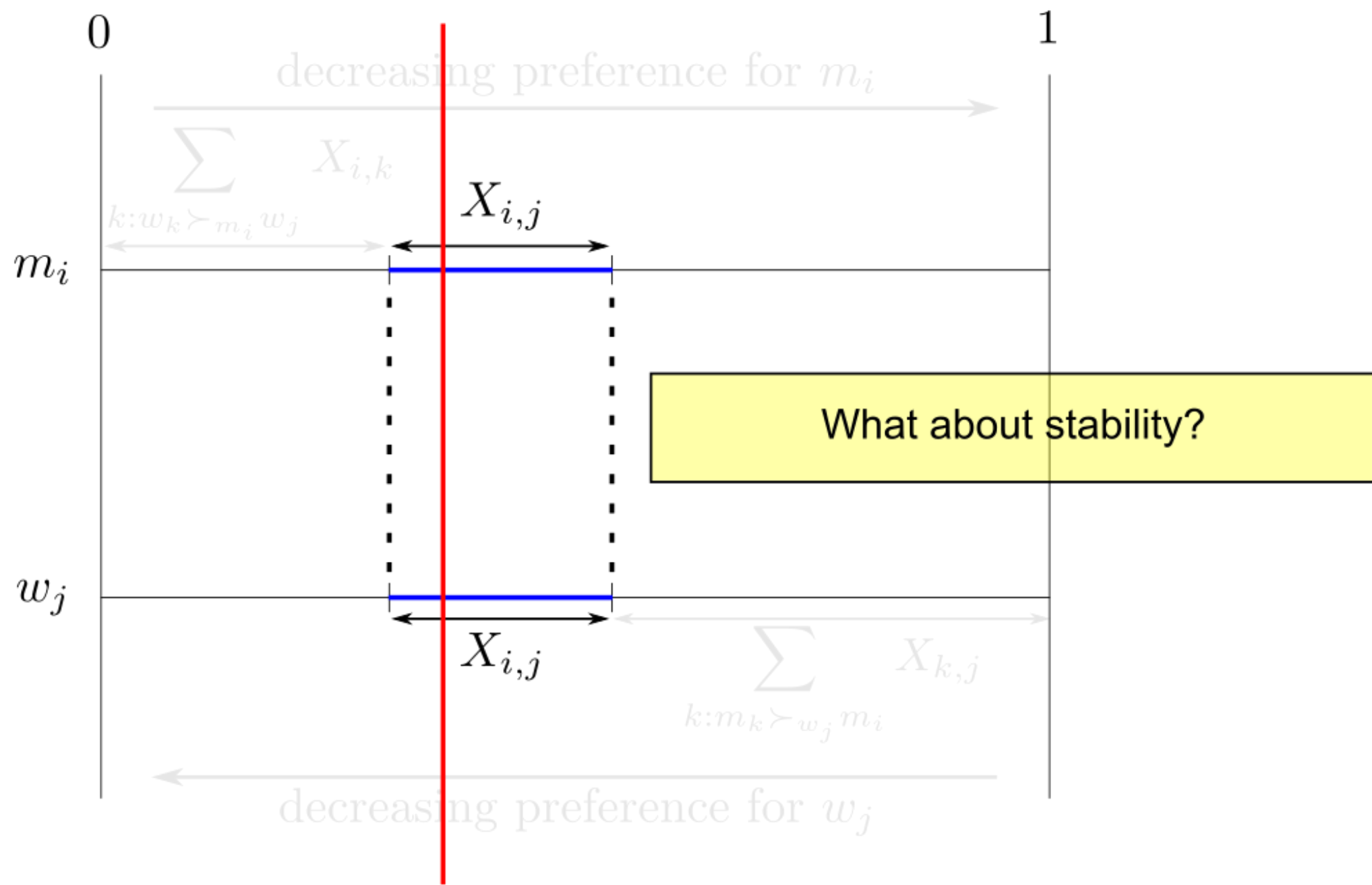


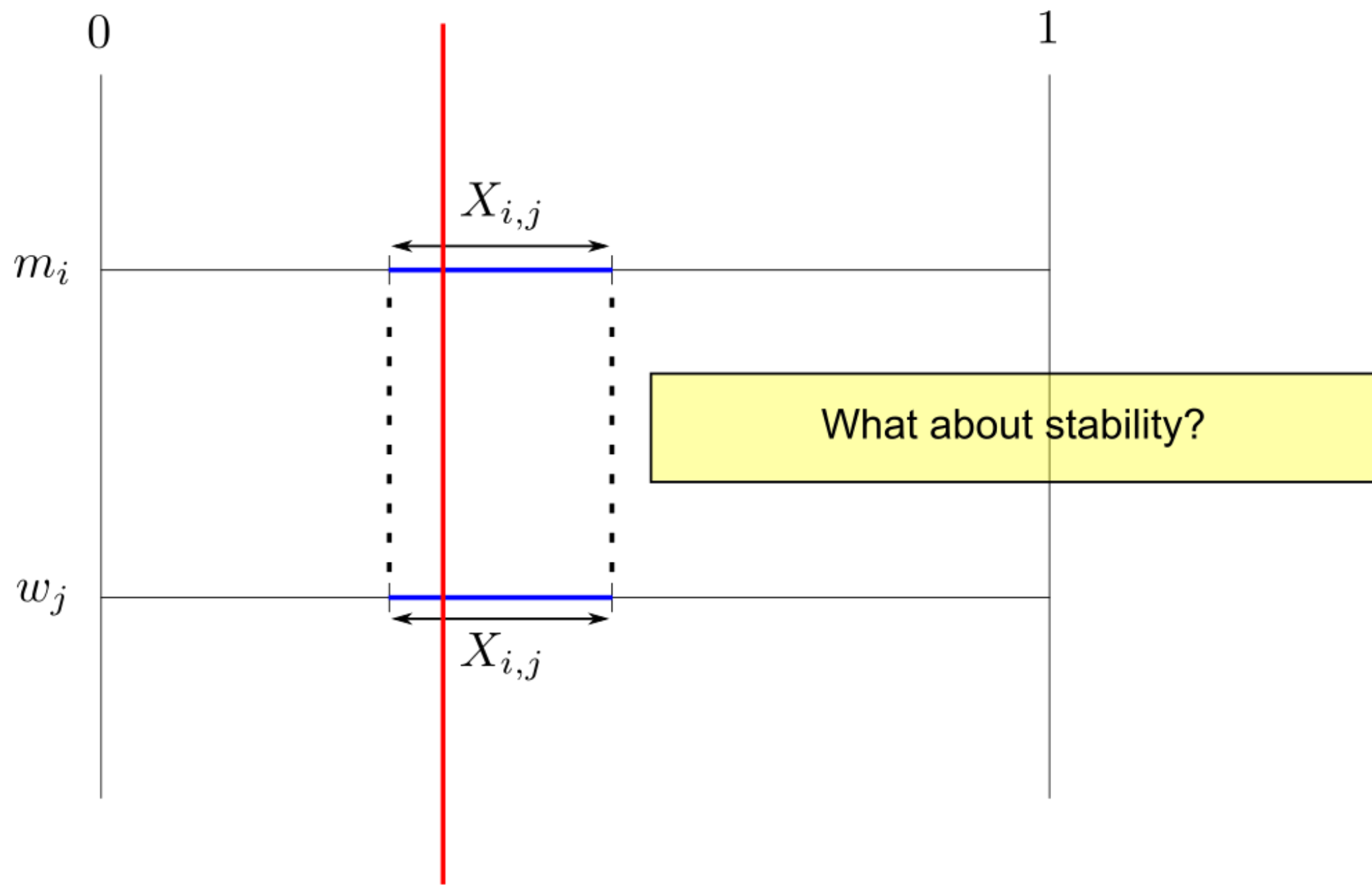


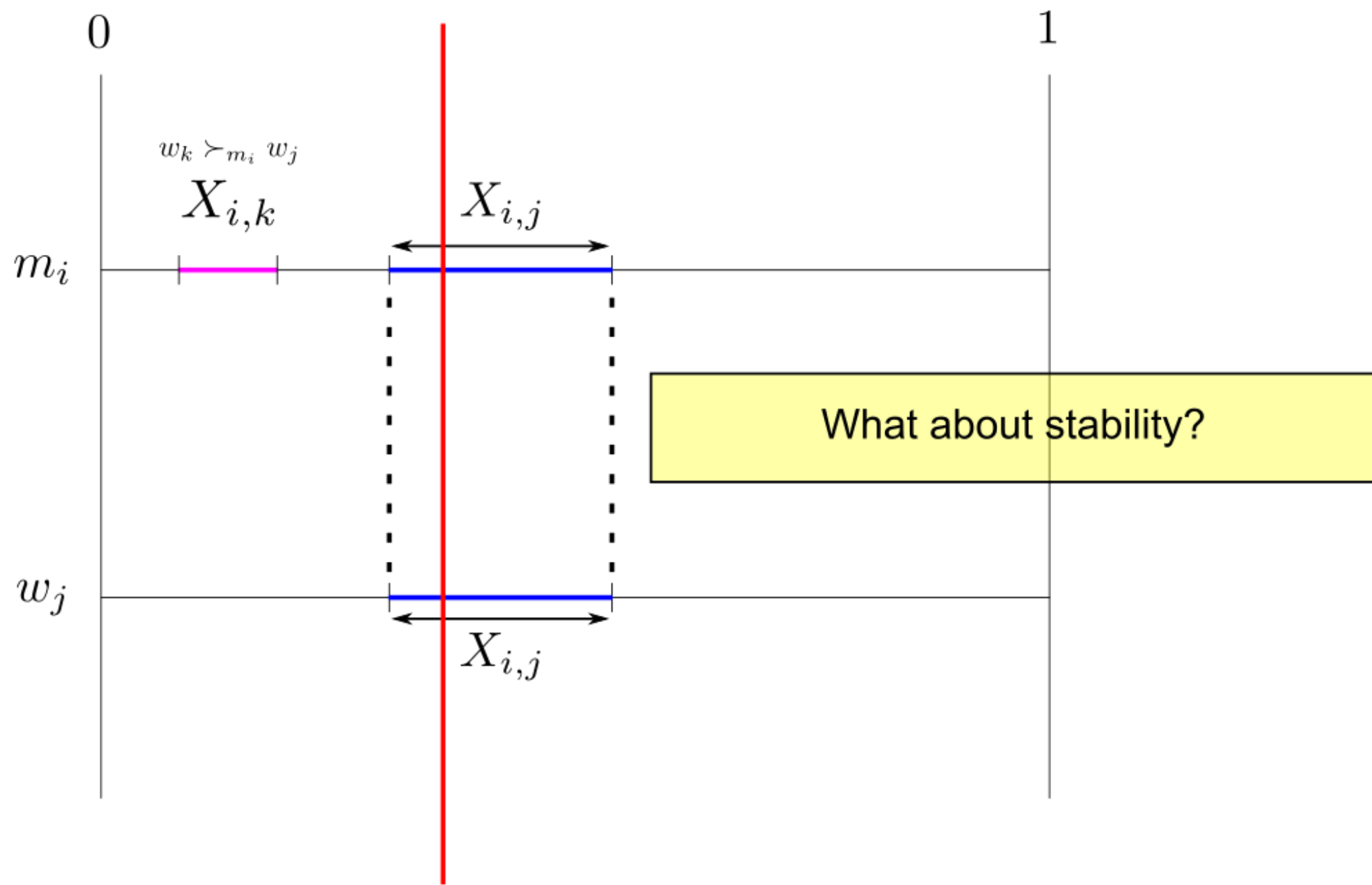


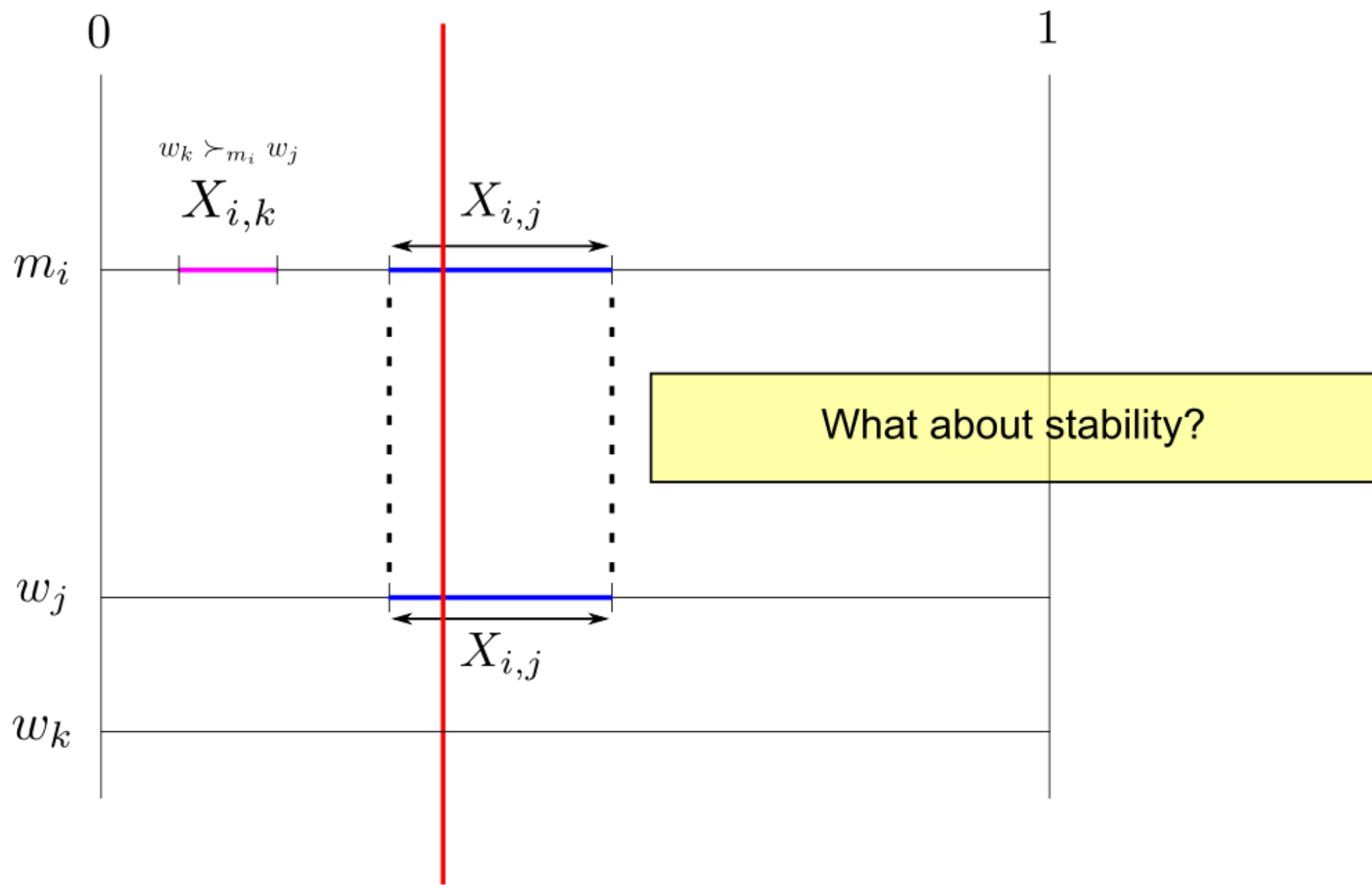


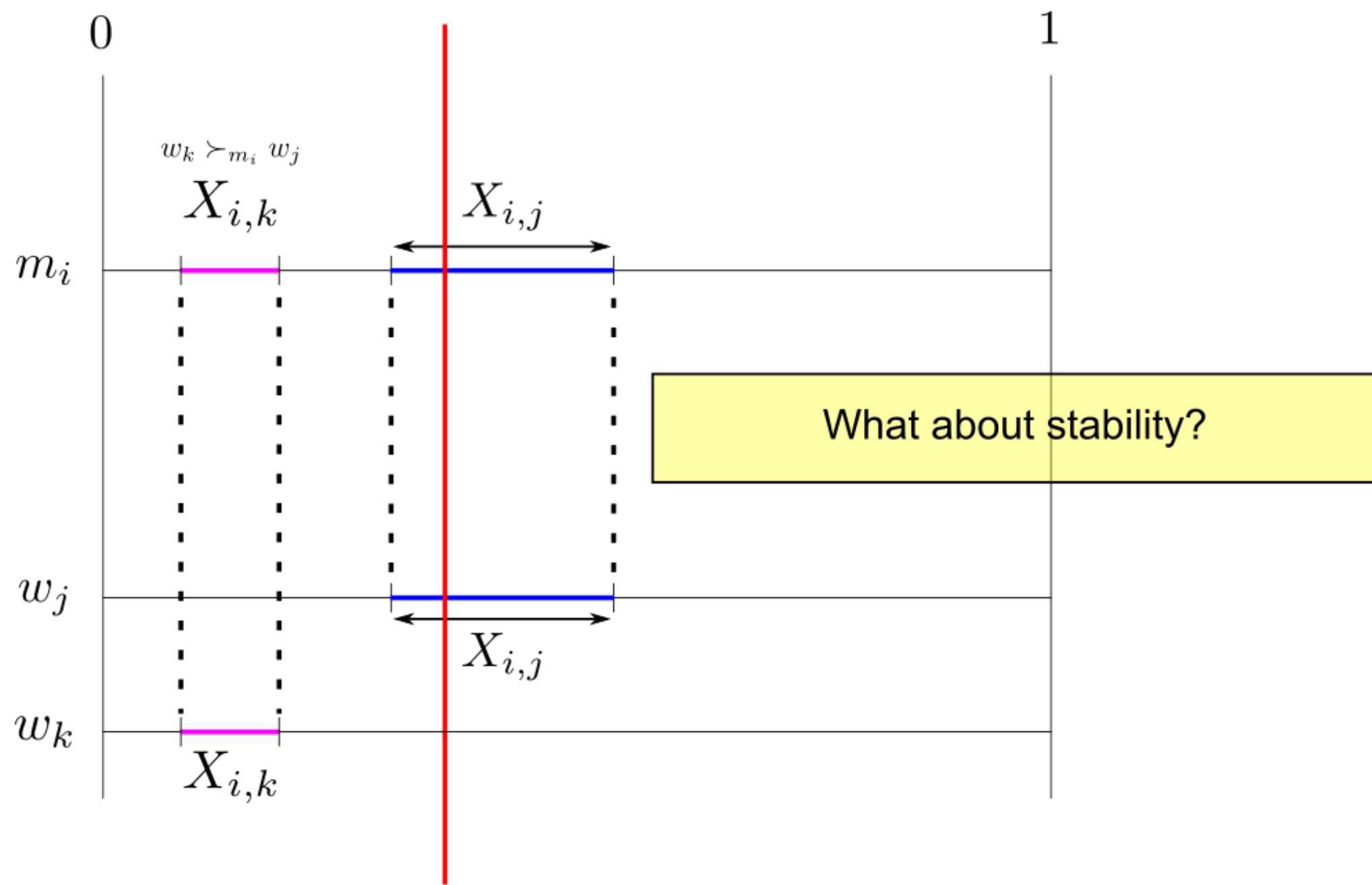


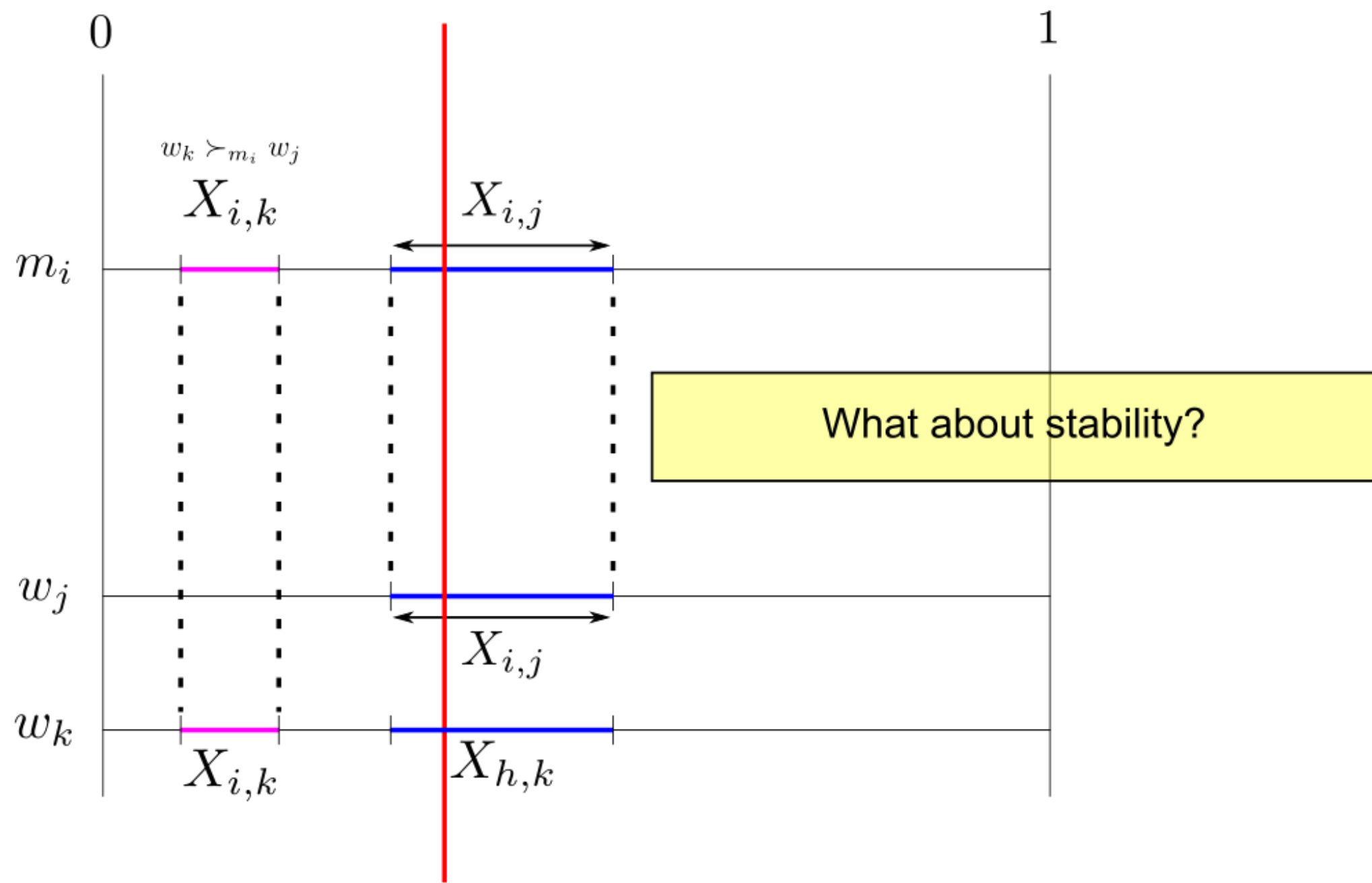


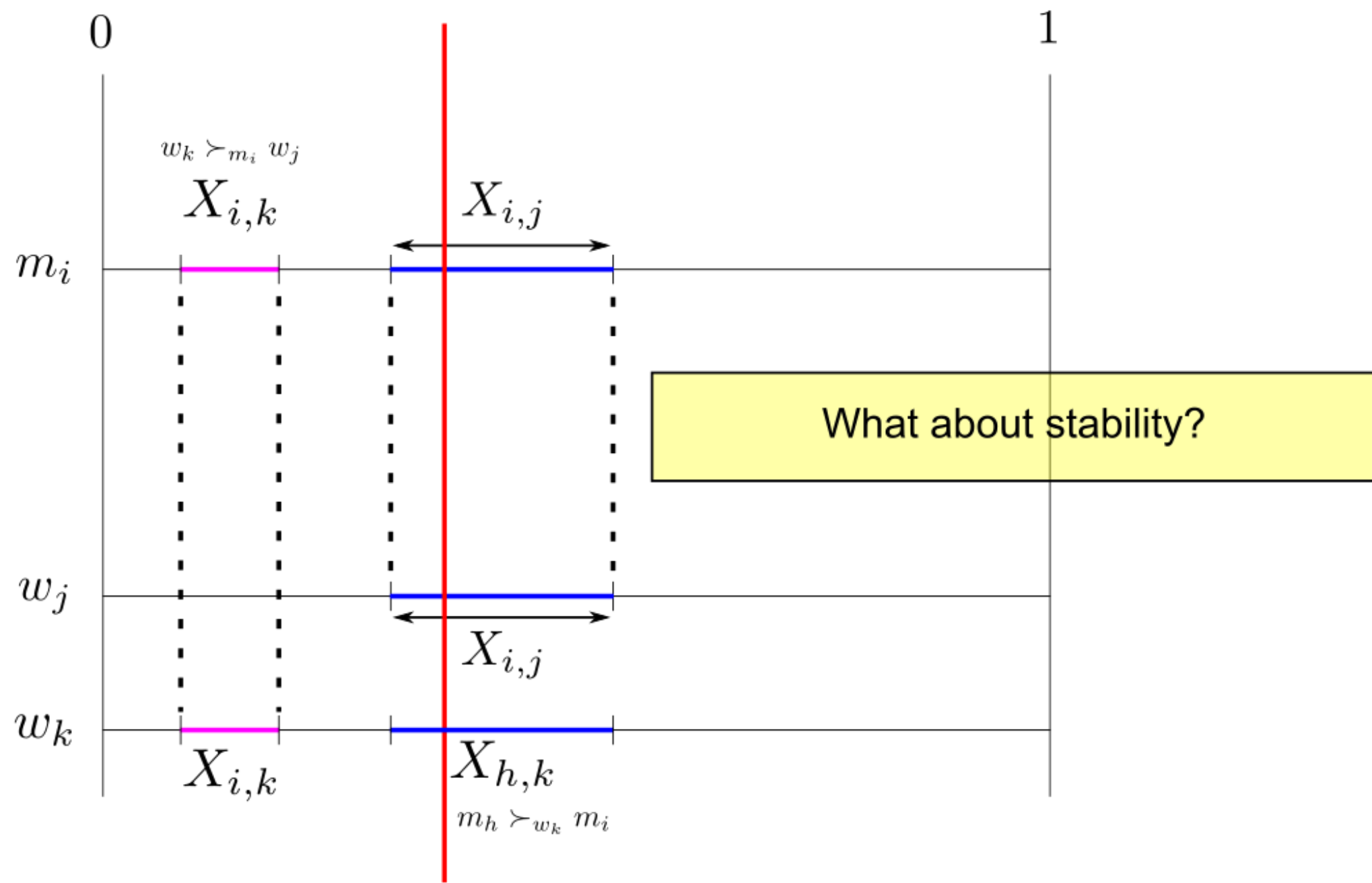


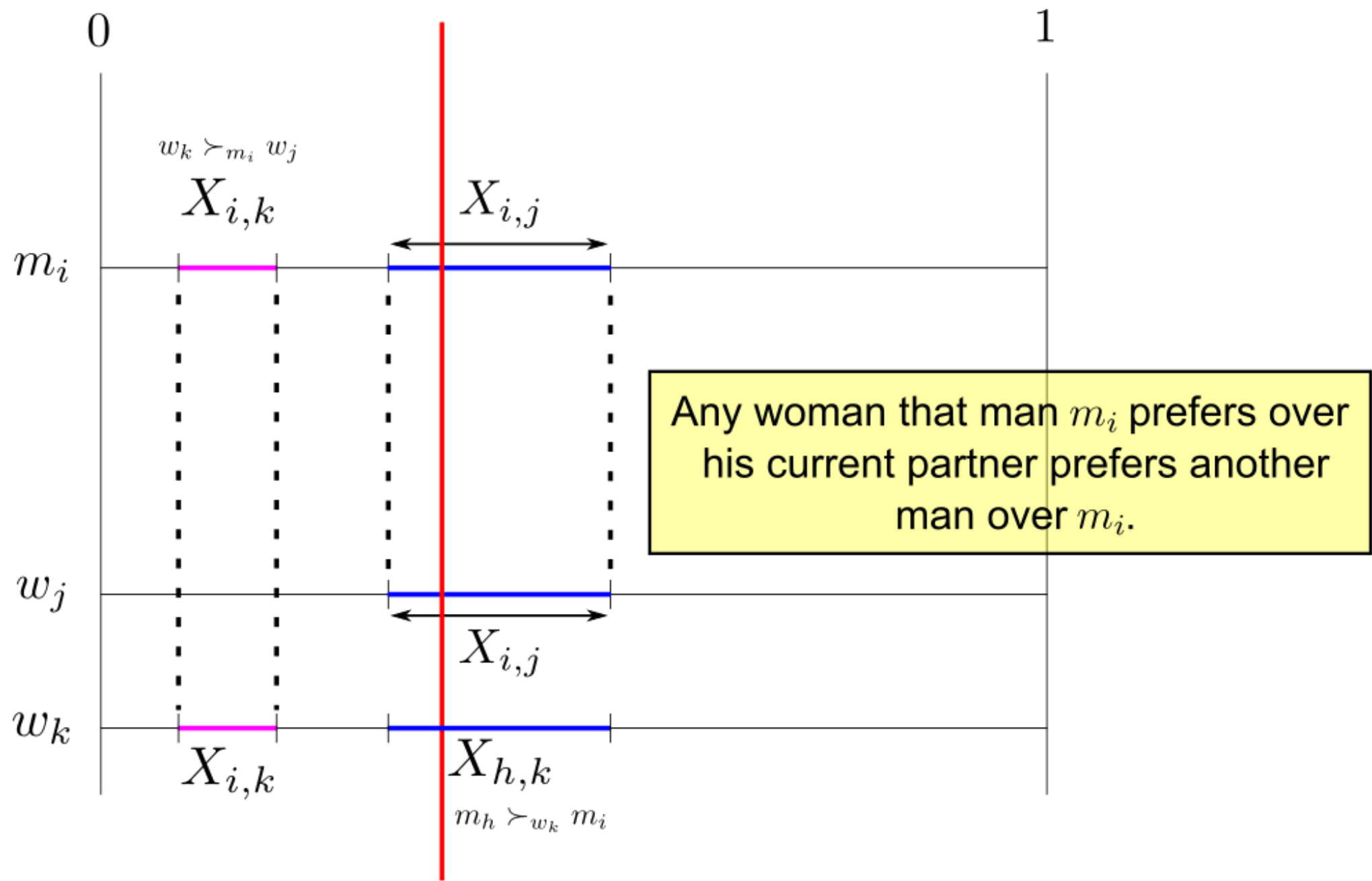


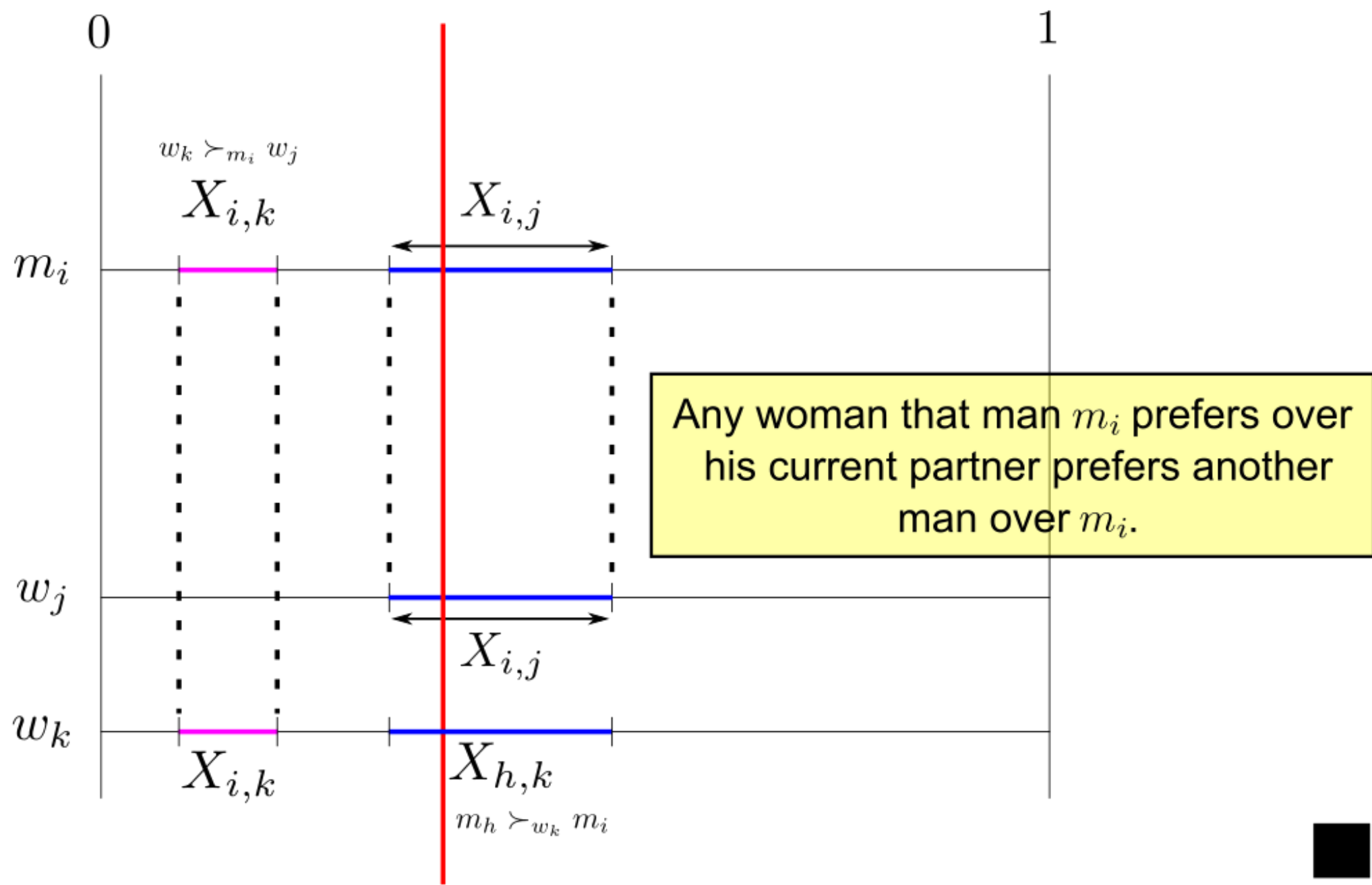












Let us use the decomposition technique to show the existence of a "fair" matching.

A "Fair" Stable Matching

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Suppose there are L stable matchings μ_1, \dots, μ_L for a given instance.

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For each man m_i , define his *median rank* as:

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Median mapping: Each agent points to its median rank agent.

1,1,1,1 | 4,4,3,3

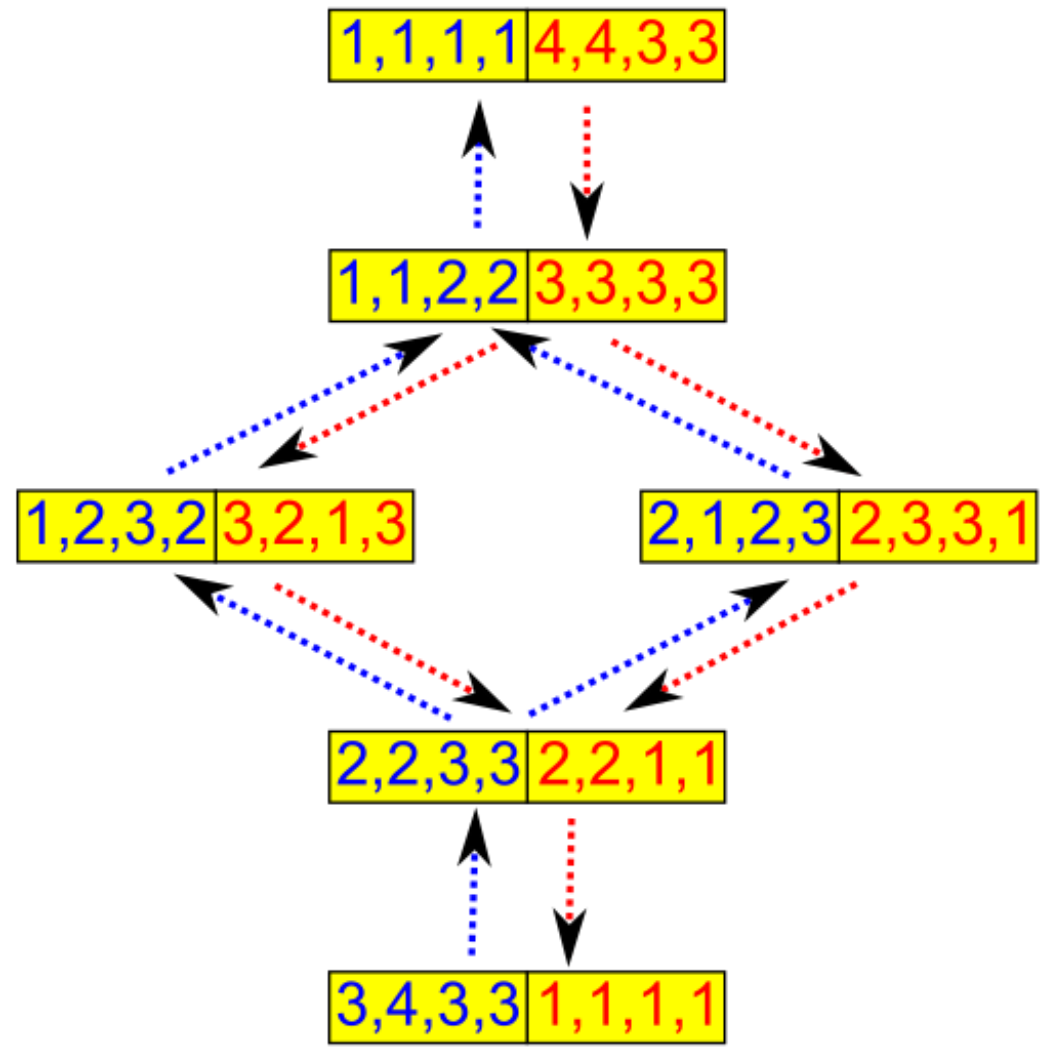
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

3,4,3,3 | 1,1,1,1



1,1,1,1 | 4,4,3,3

m_1 1 1 1 2 2 3

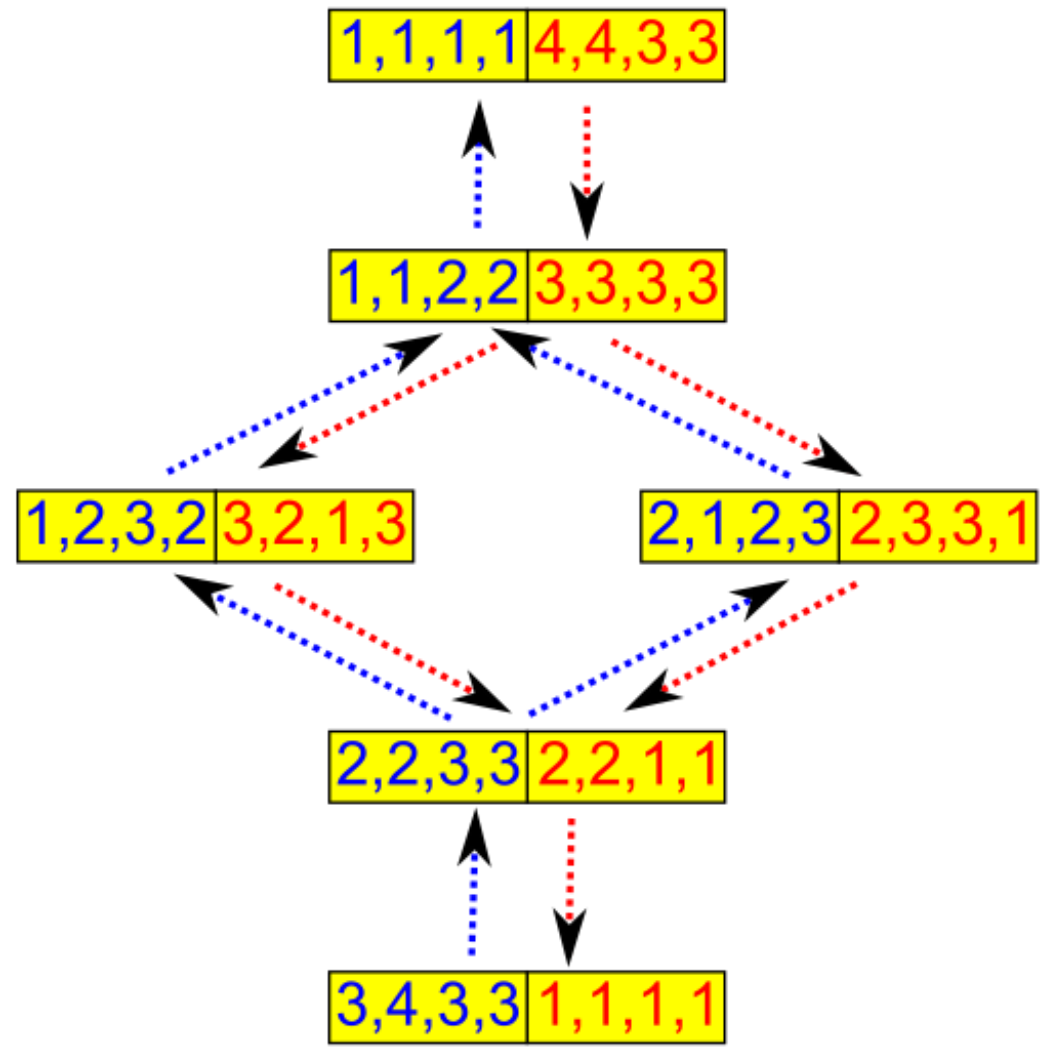
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

3,4,3,3 | 1,1,1,1



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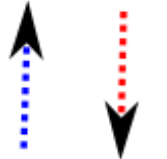
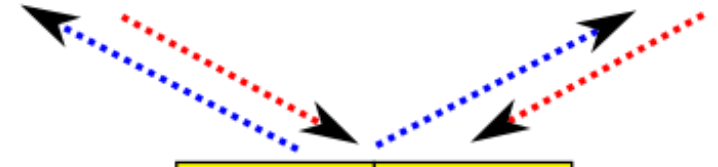
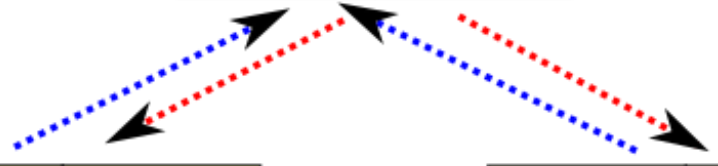
1,2,3,2 | 3,2,1,3

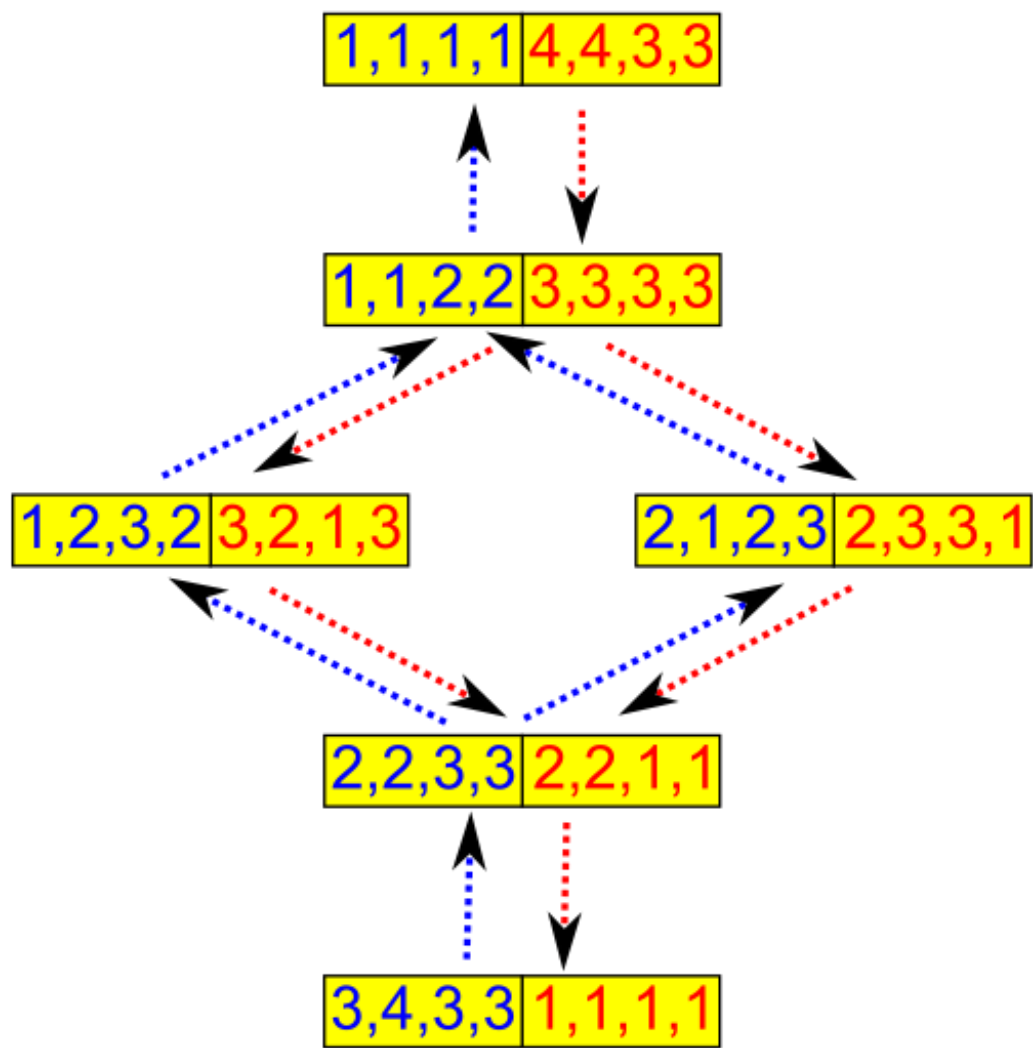
2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

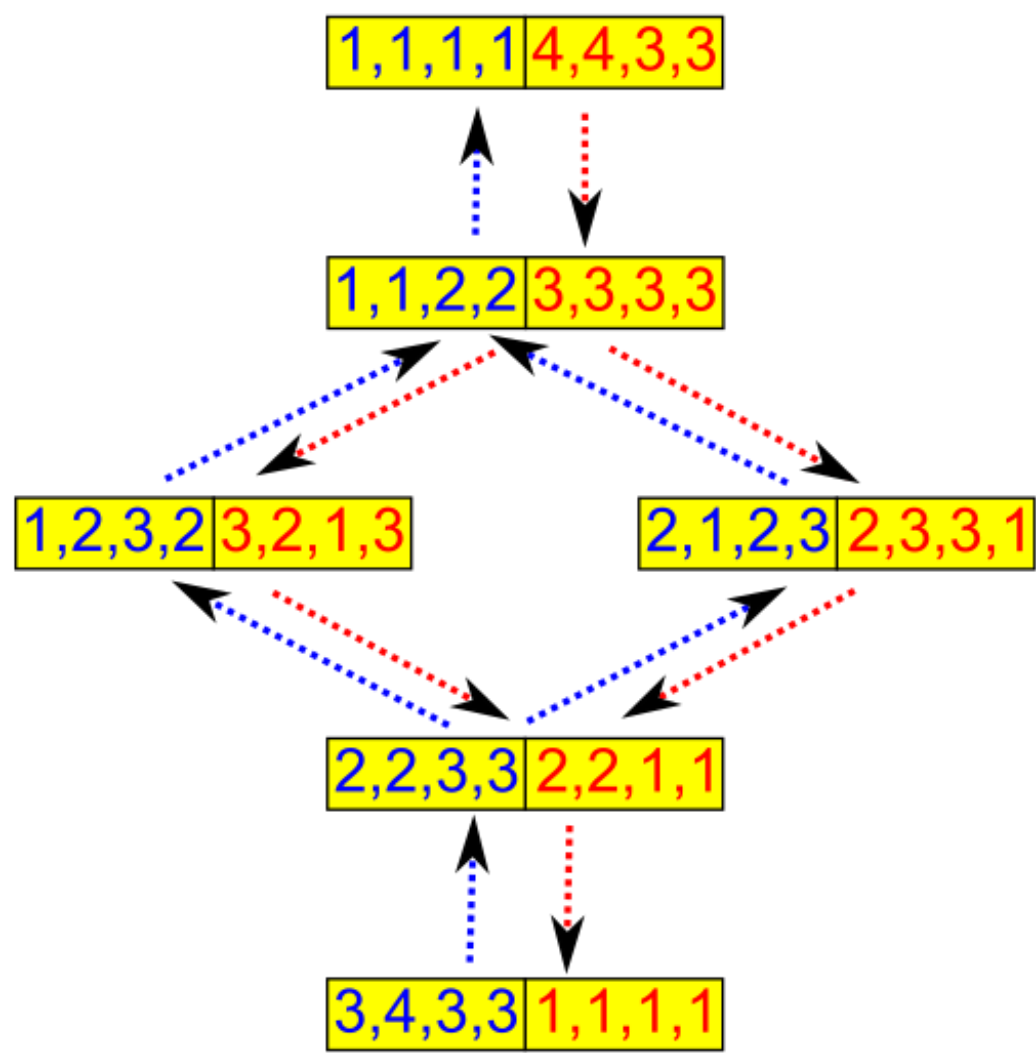
3,4,3,3 | 1,1,1,1

m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4

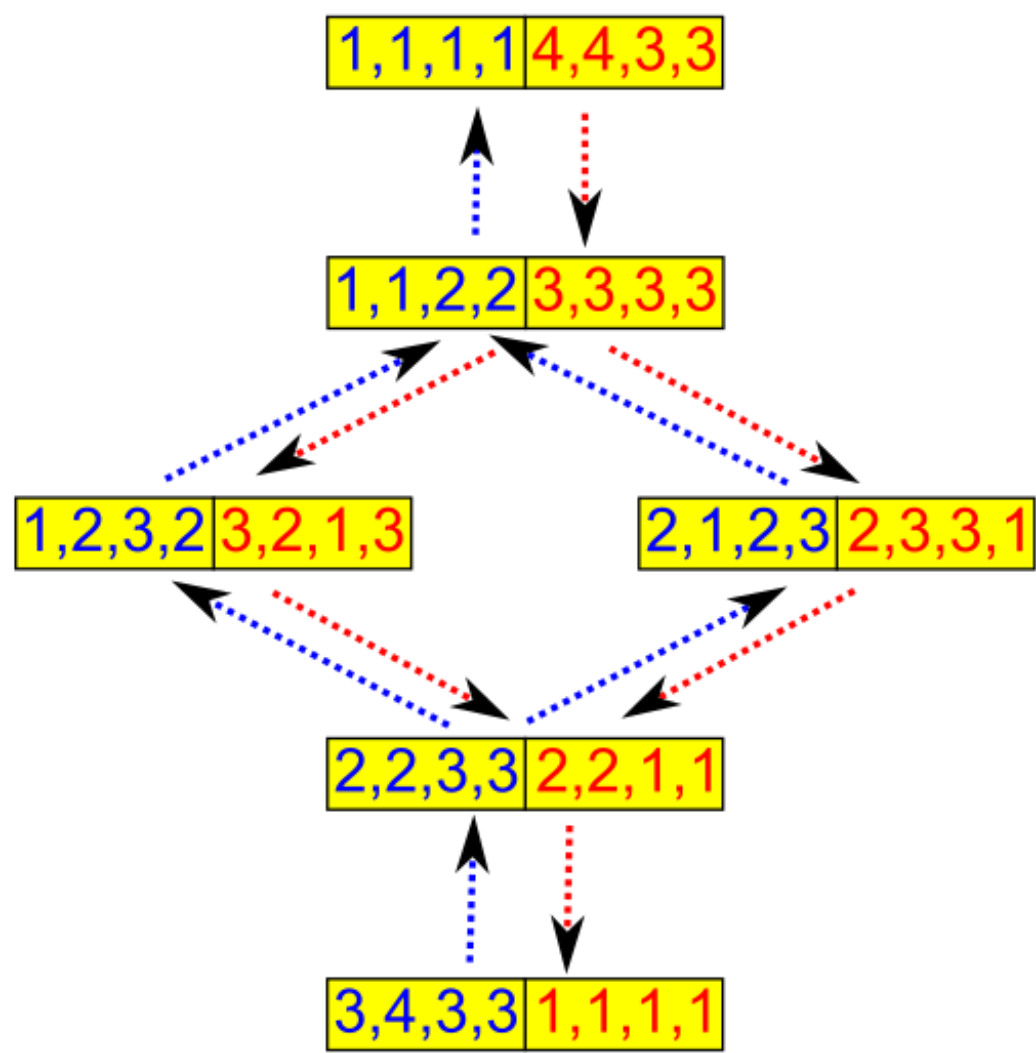




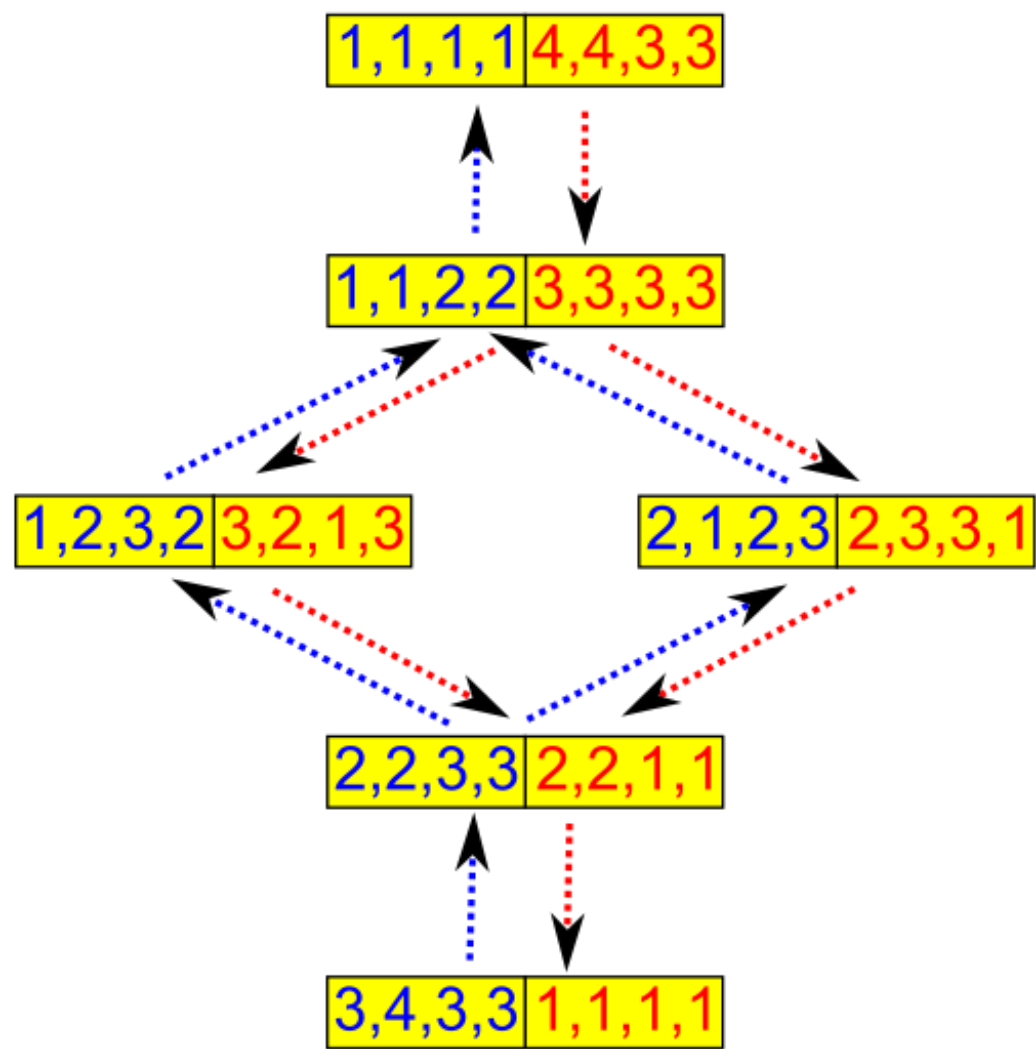
m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1



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m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1

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[Teo and Sethuraman, *MOR* 1998]

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Proof by example.

1,1,1,1 | 4,4,3,3

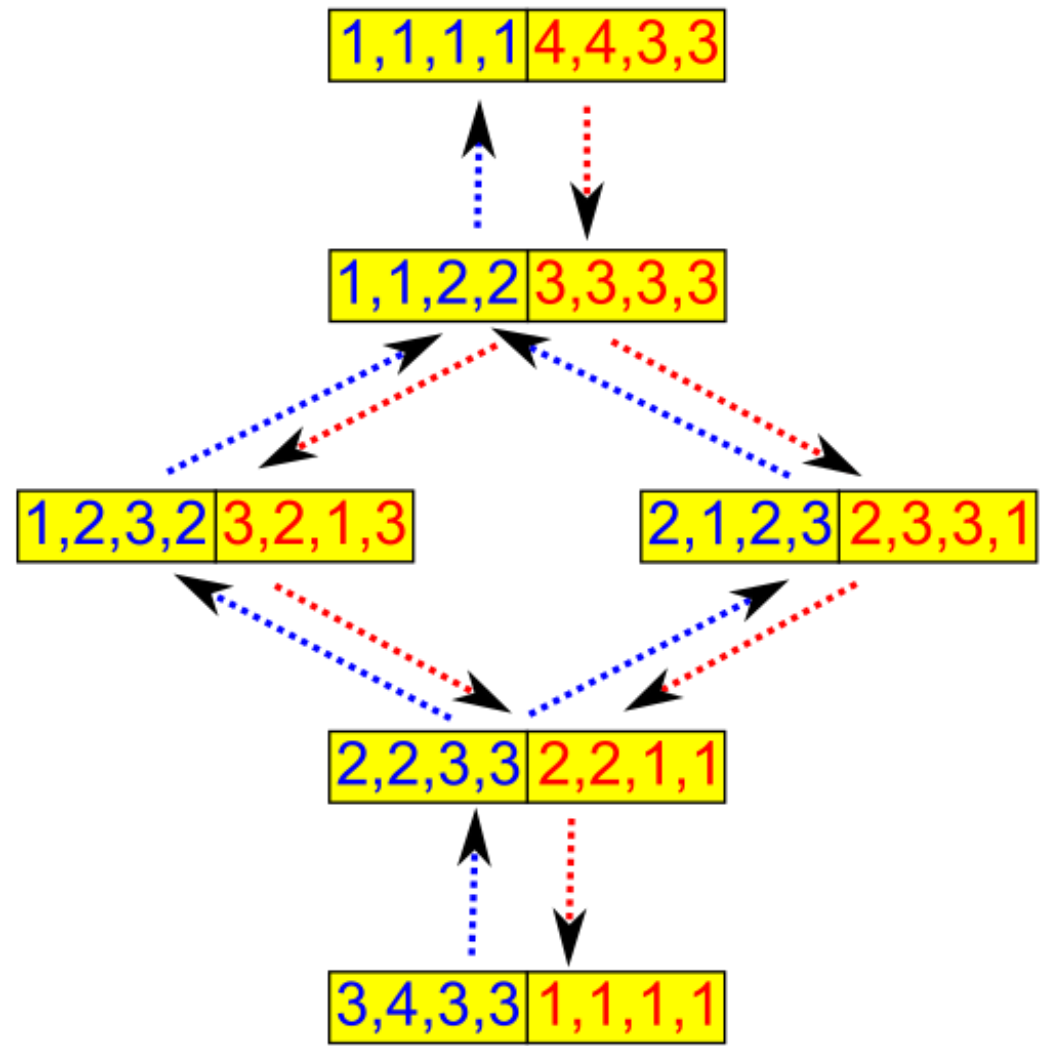
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

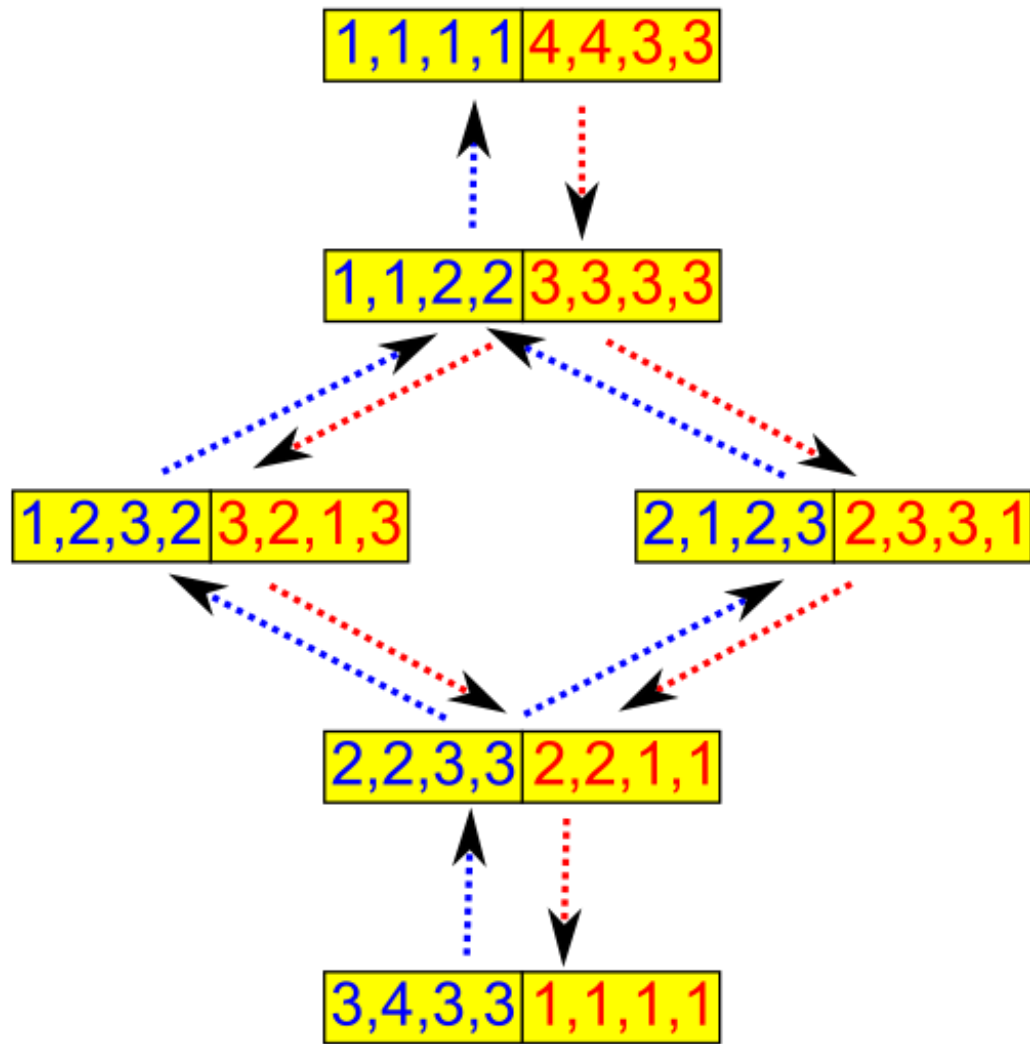
2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

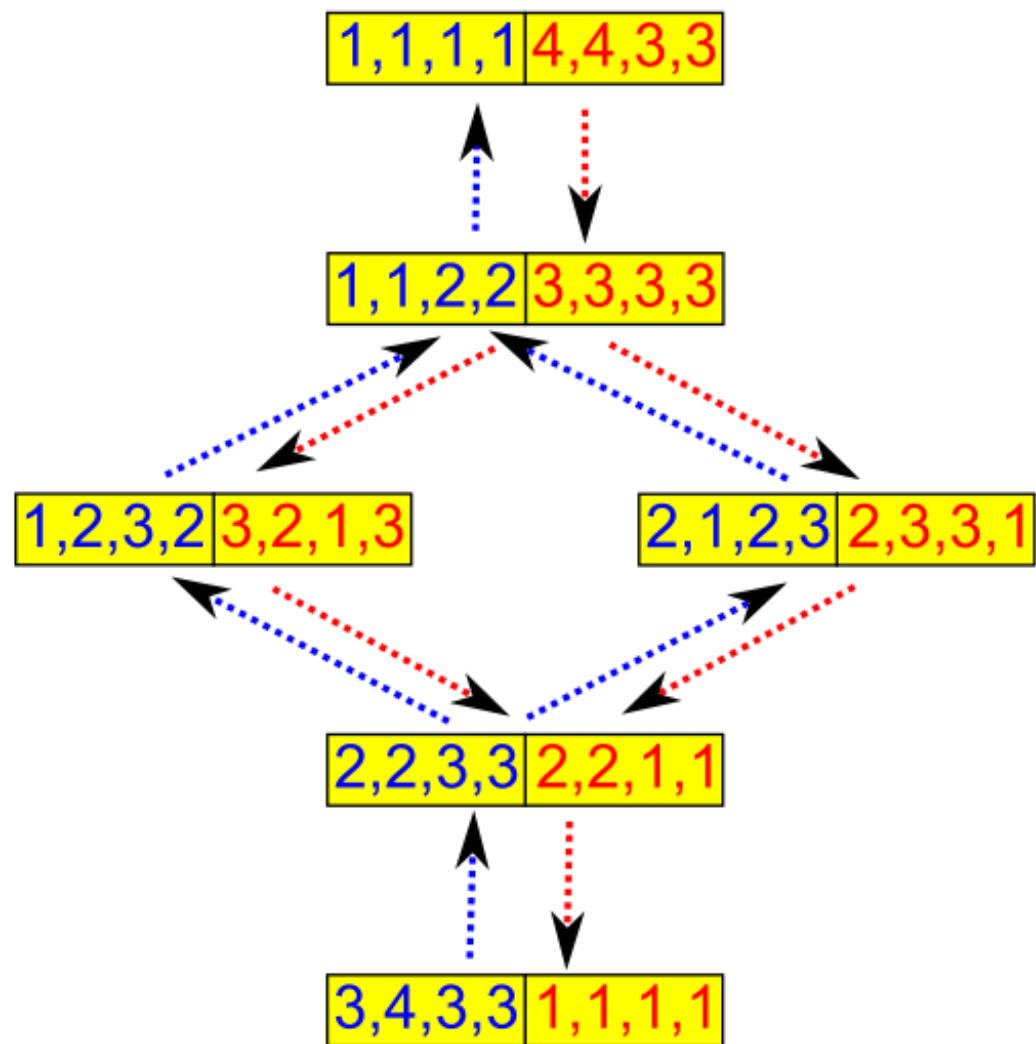
3,4,3,3 | 1,1,1,1



Consider a uniform combination of all integral stable matchings.



Consider a uniform combination of all integral stable matchings.



m_1

1

1

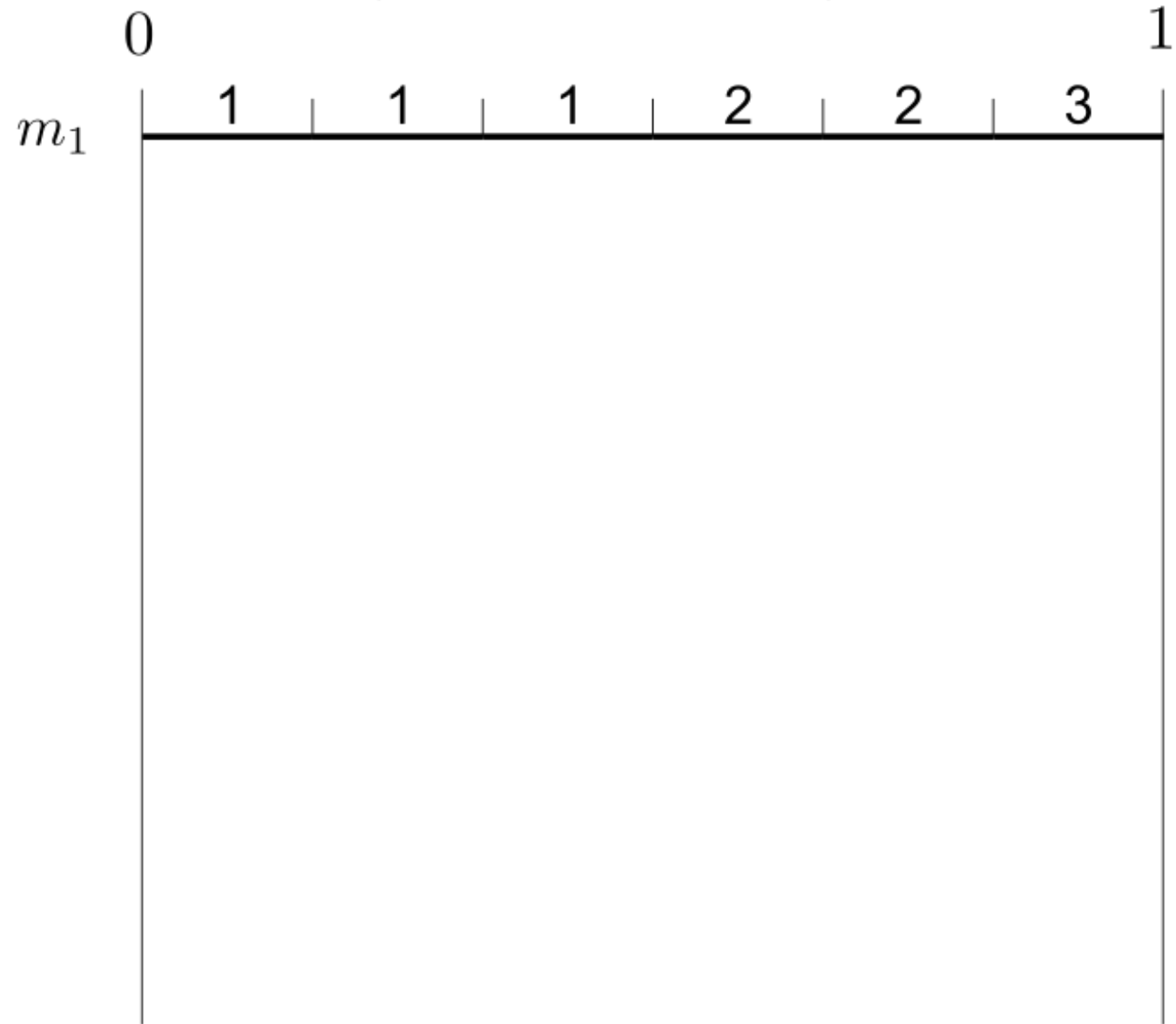
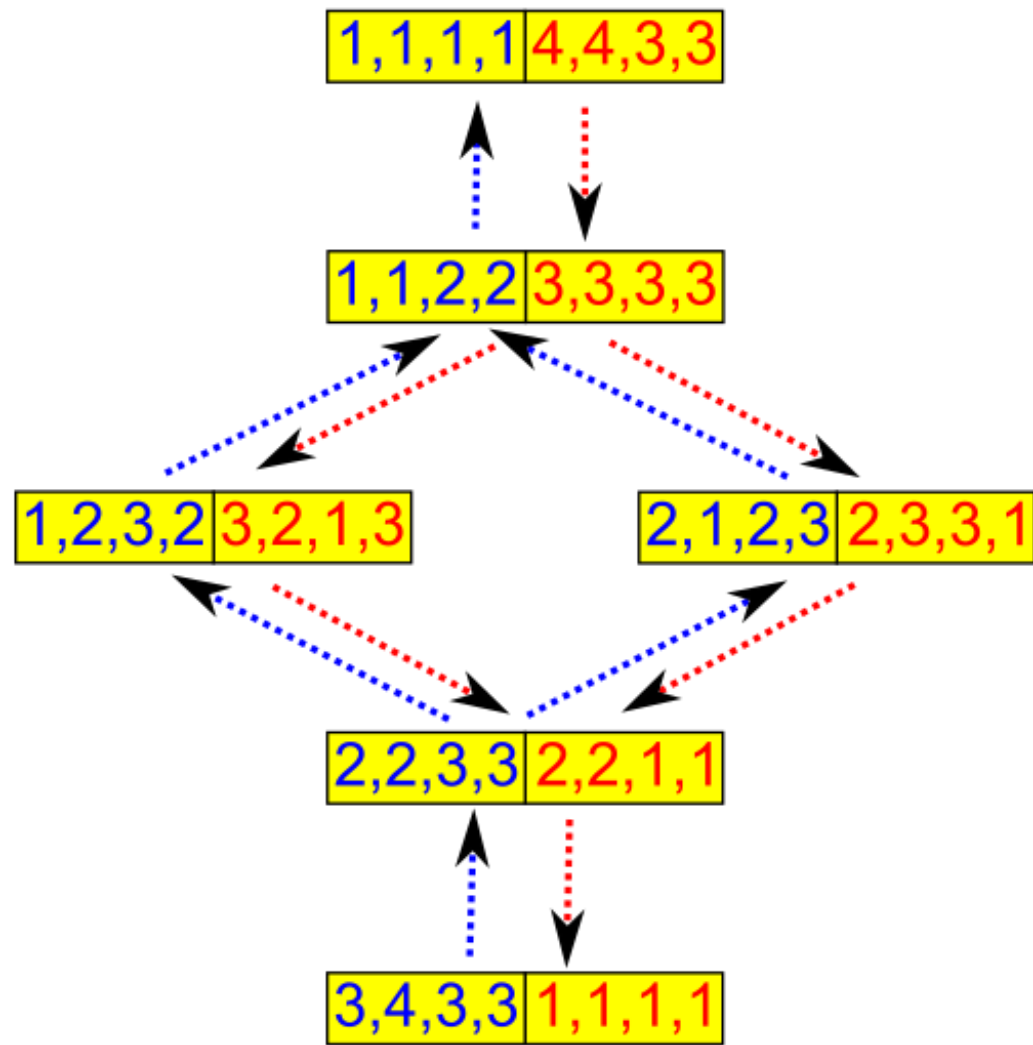
1

2

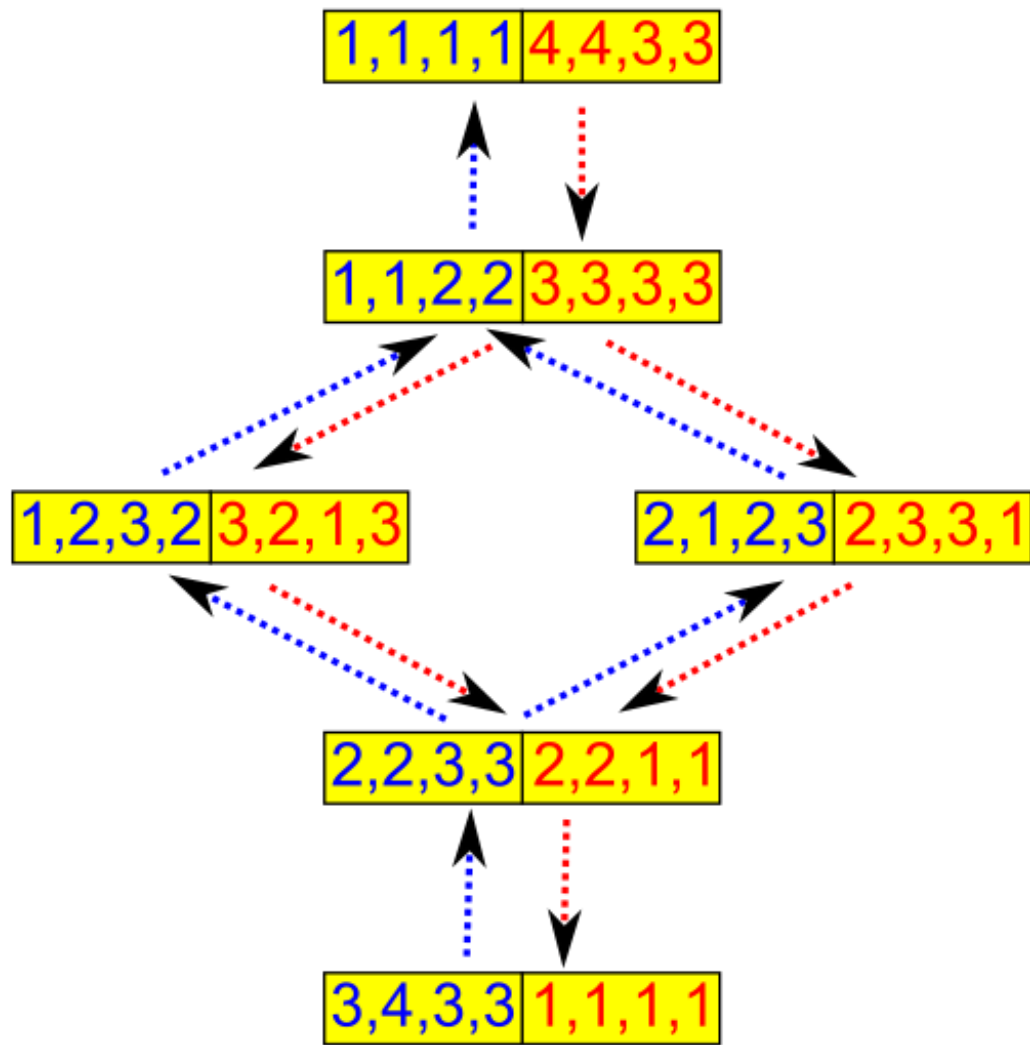
2

3

Consider a uniform combination of all integral stable matchings.

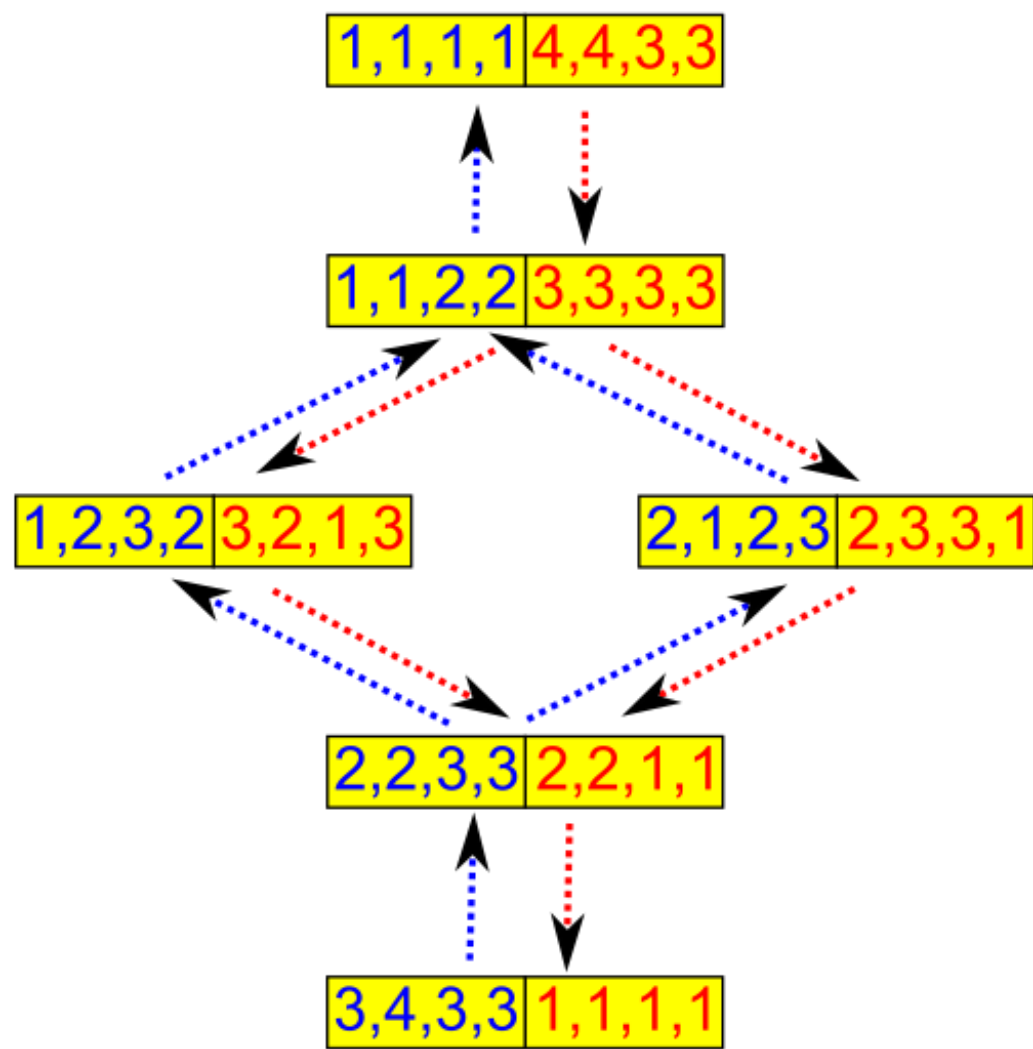


Consider a uniform combination of all integral stable matchings.



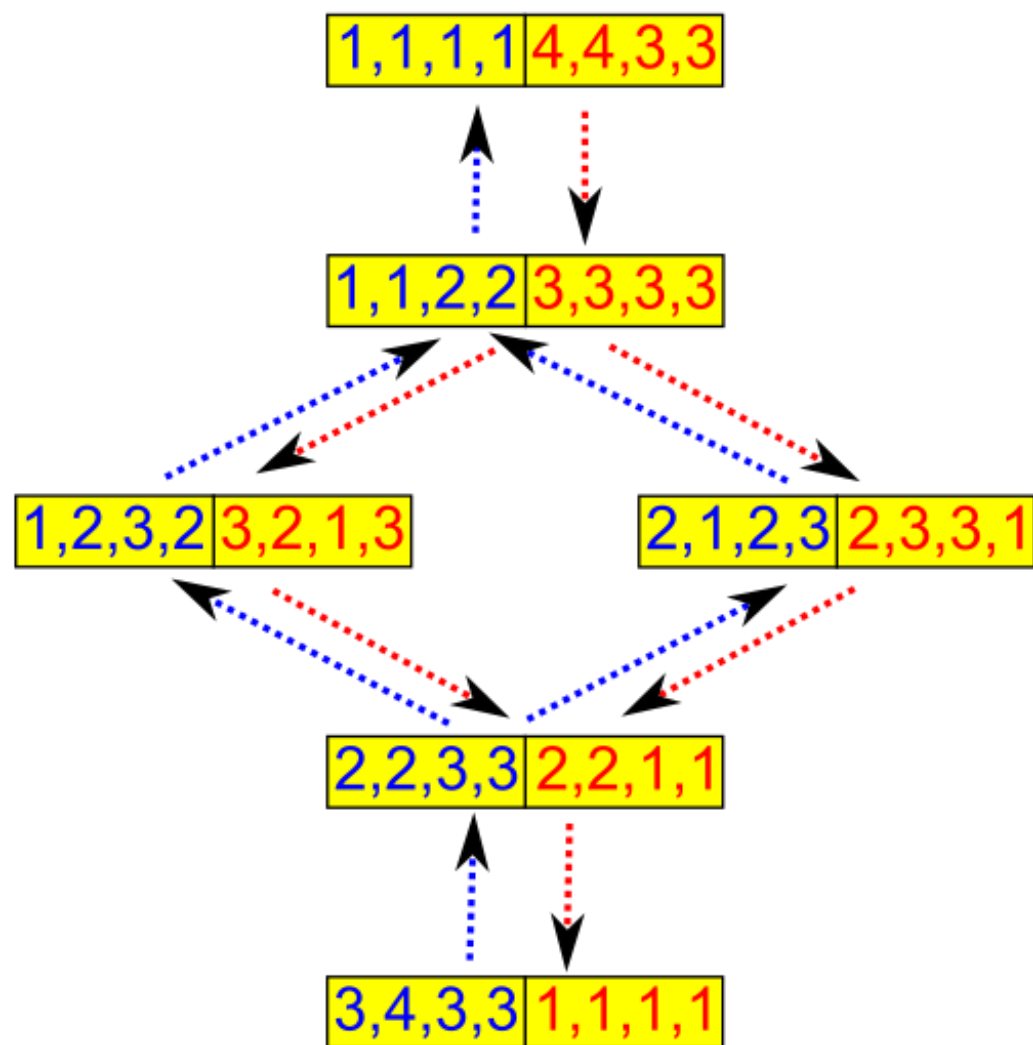
	0						1
m_1	1	1	1	2	2	3	
m_2	1	1	1	2	2	4	

Consider a uniform combination of all integral stable matchings.



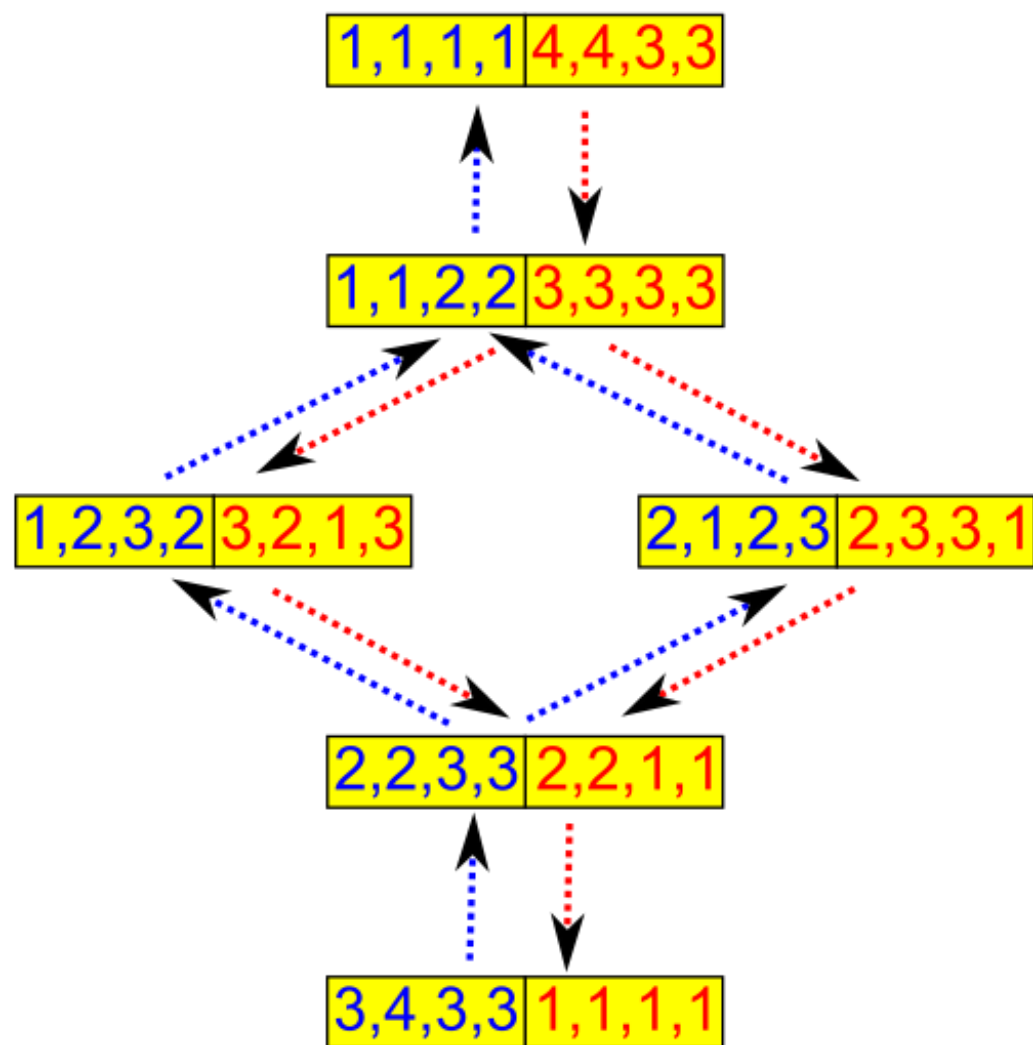
	0					1
m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1

Consider a uniform combination of all integral stable matchings.



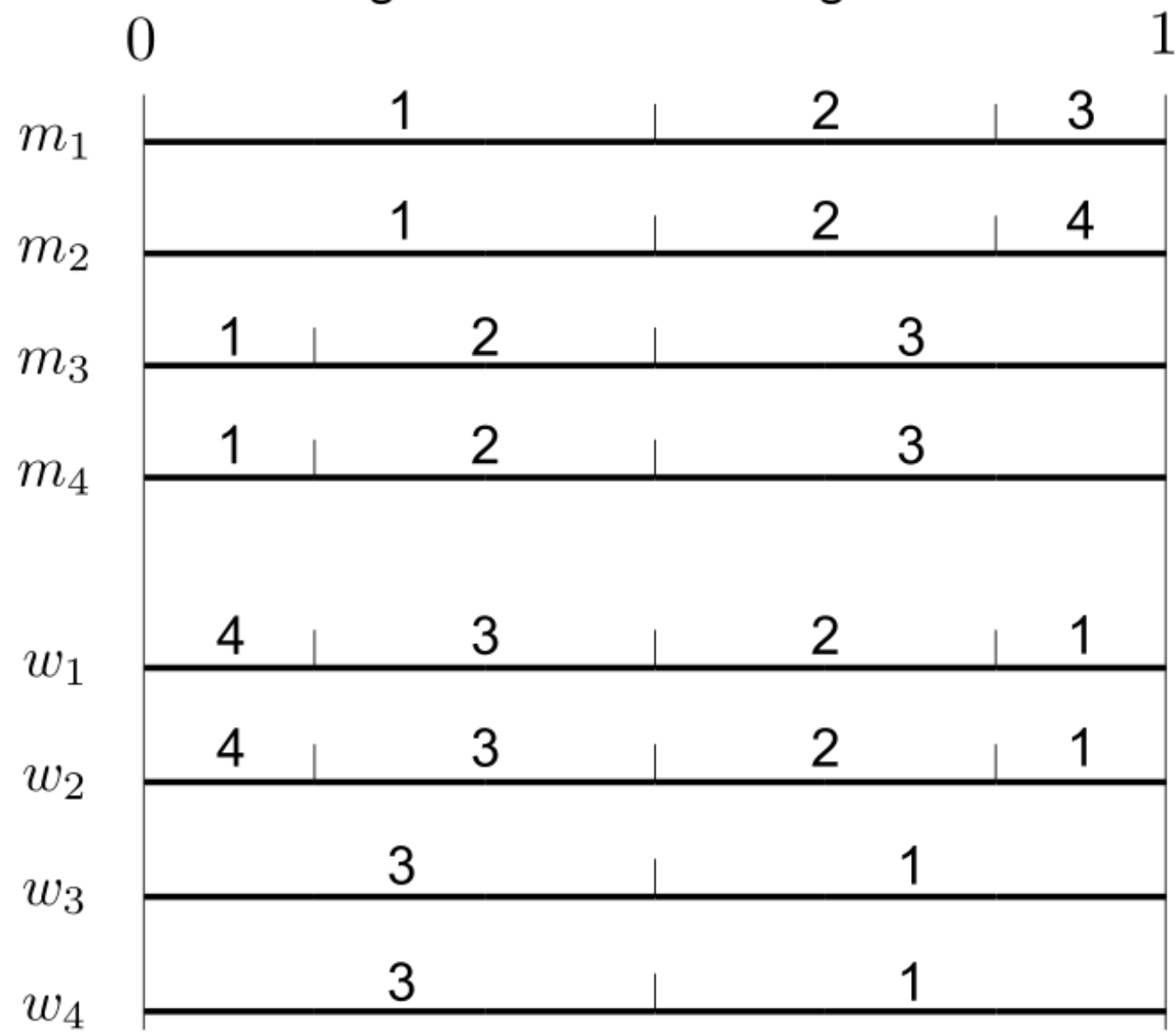
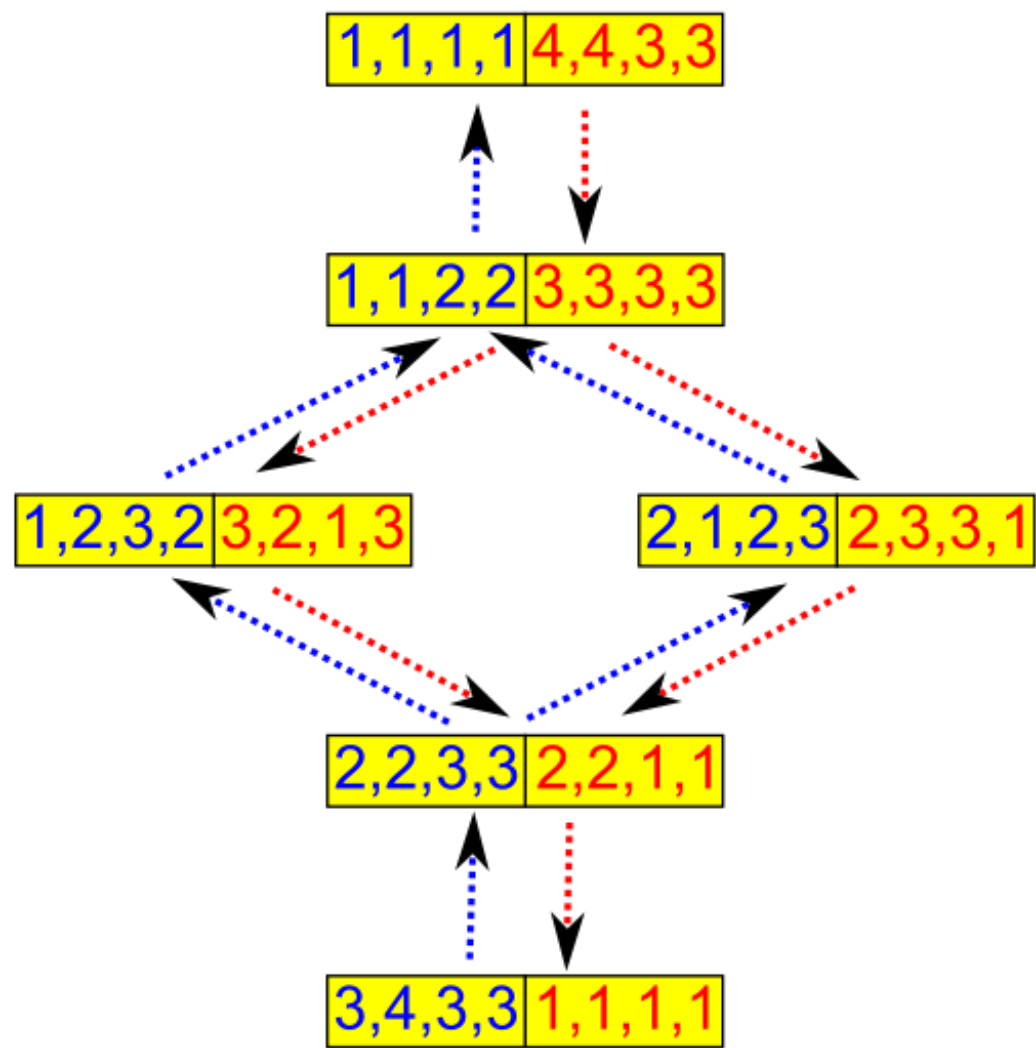
	0					1
m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1

Consider a uniform combination of all integral stable matchings.

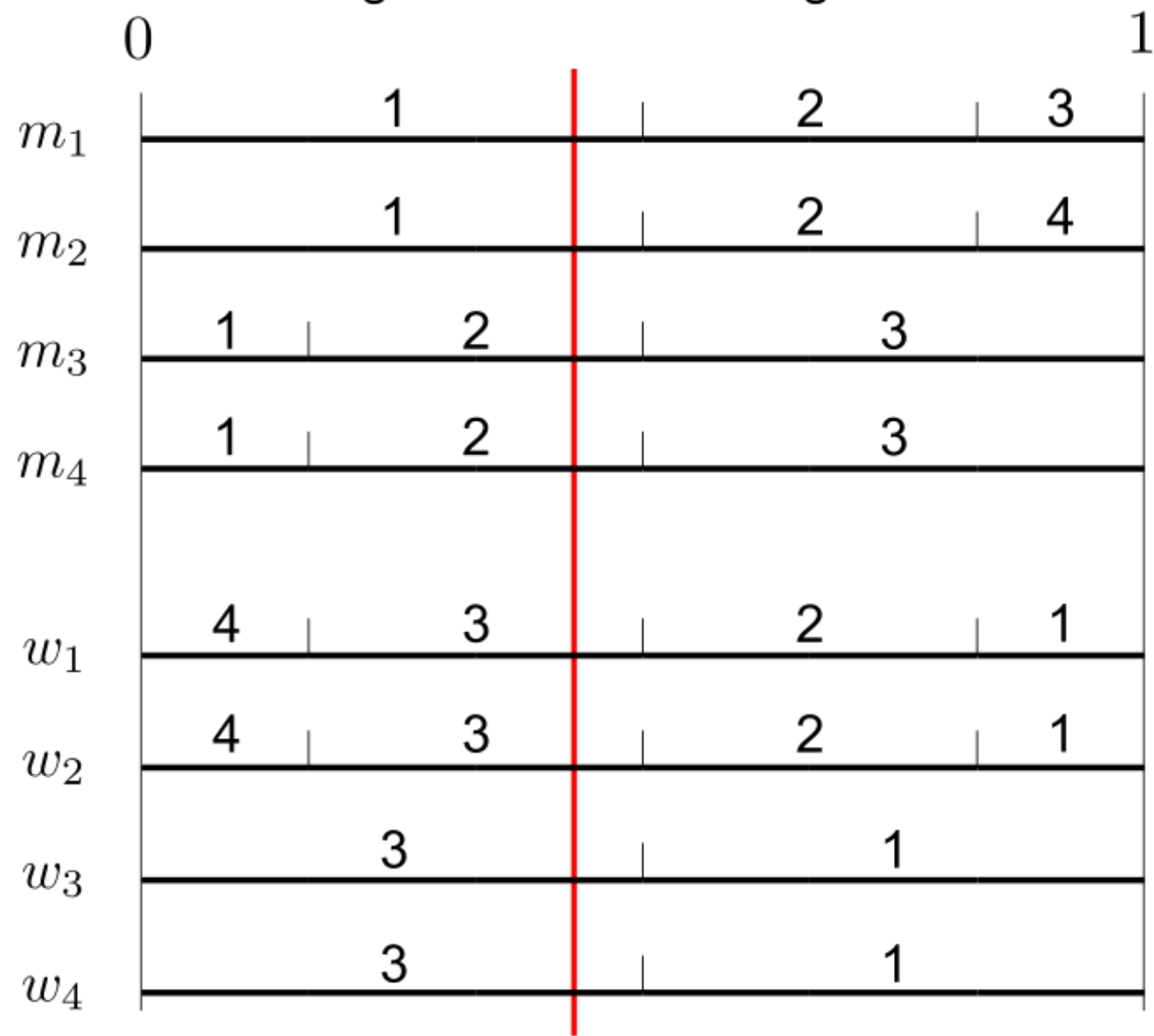
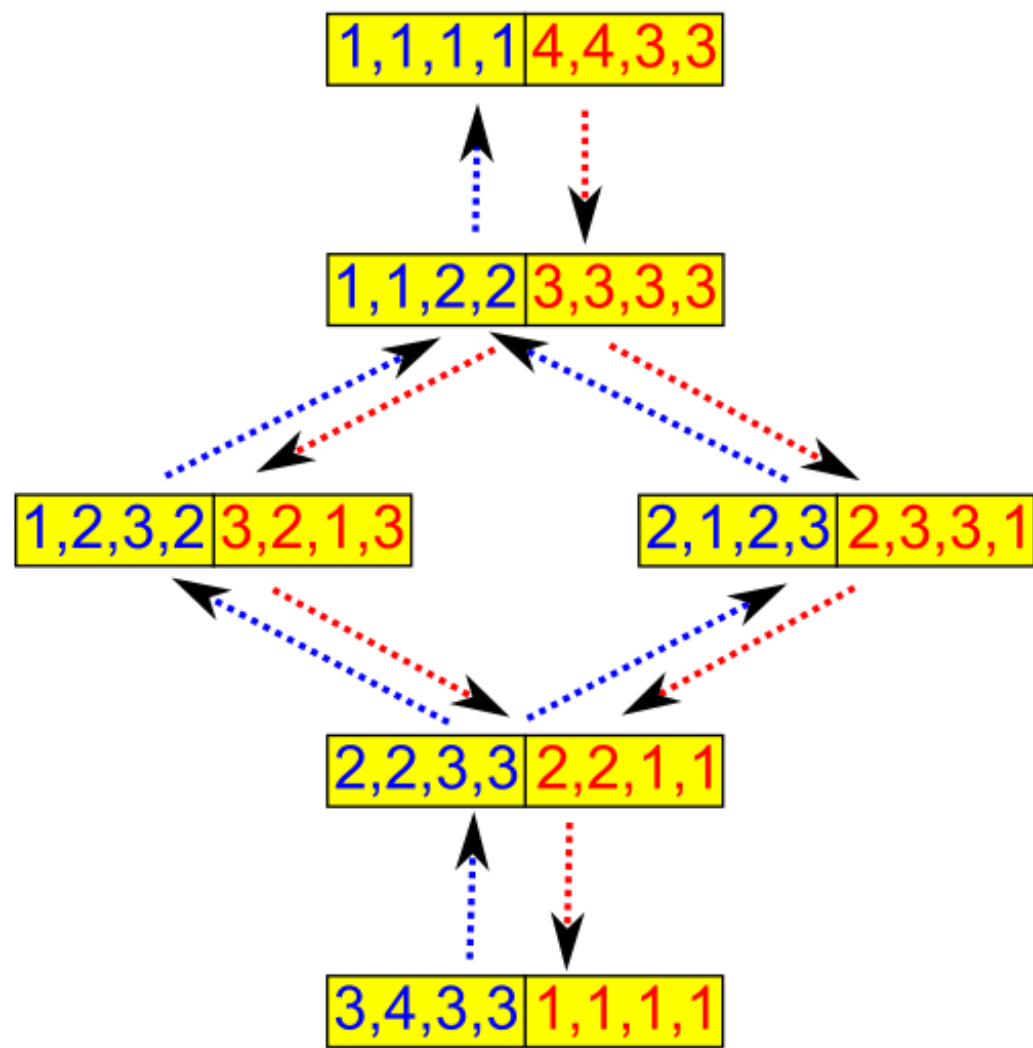


	0					1
m_1		1		2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1

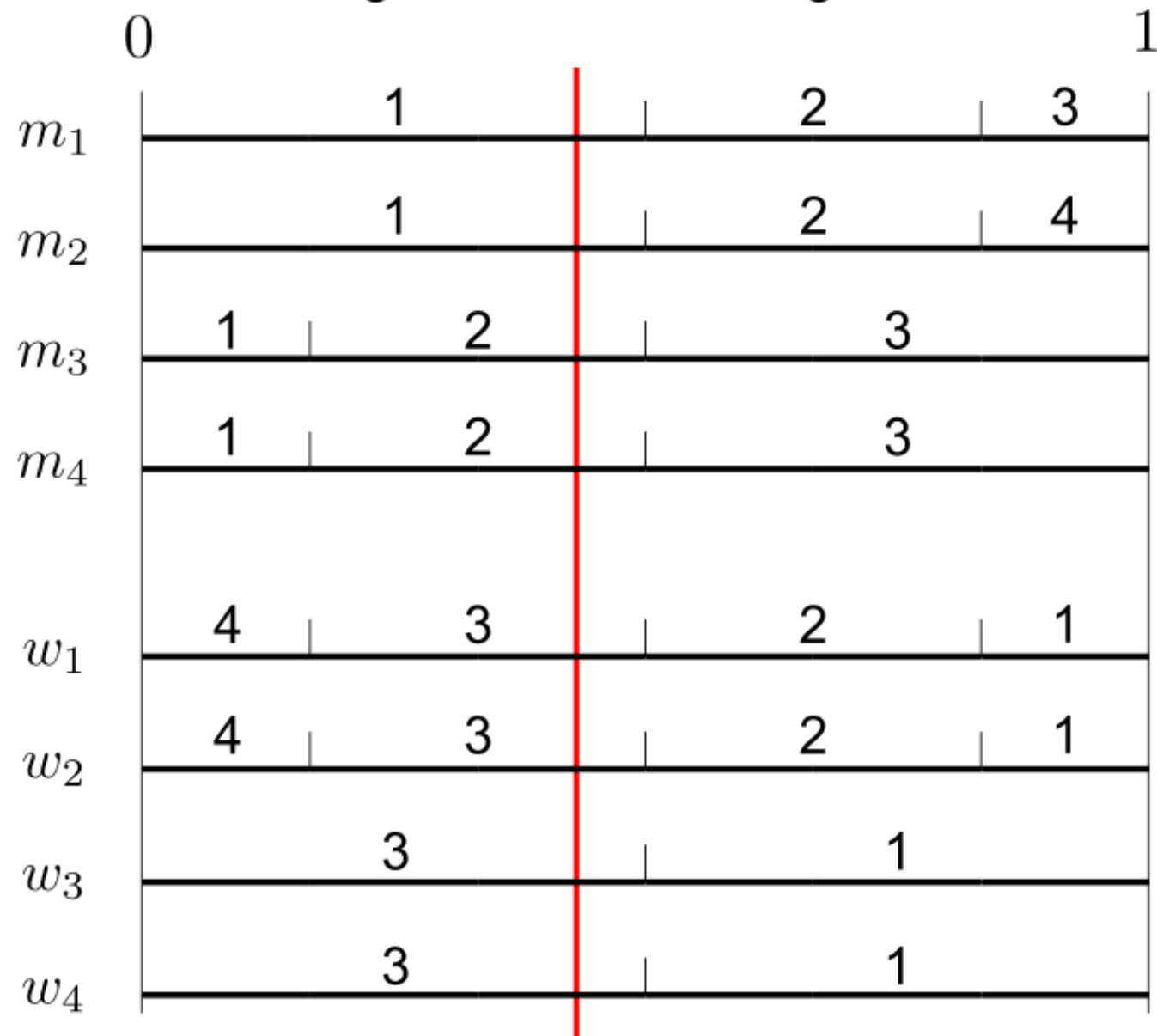
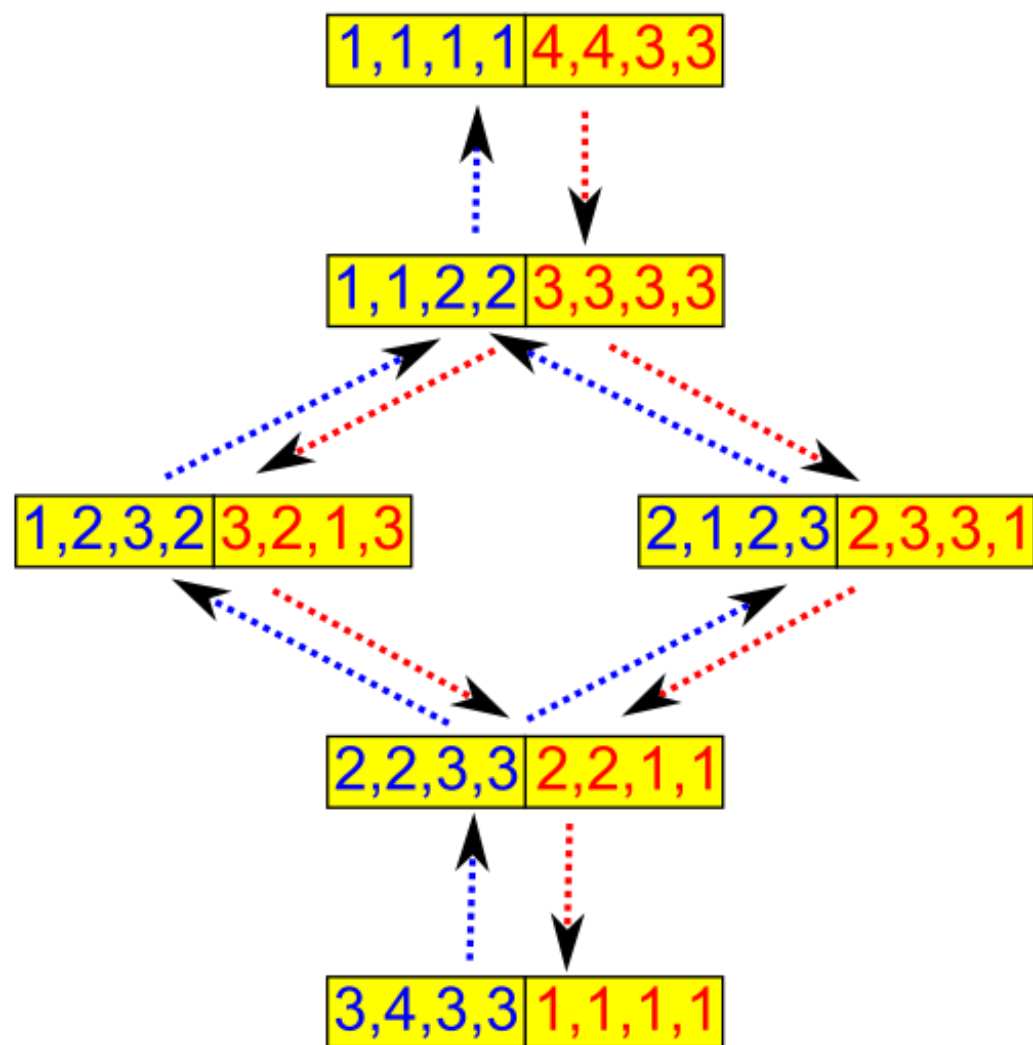
Consider a uniform combination of all integral stable matchings.



Consider a uniform combination of all integral stable matchings.

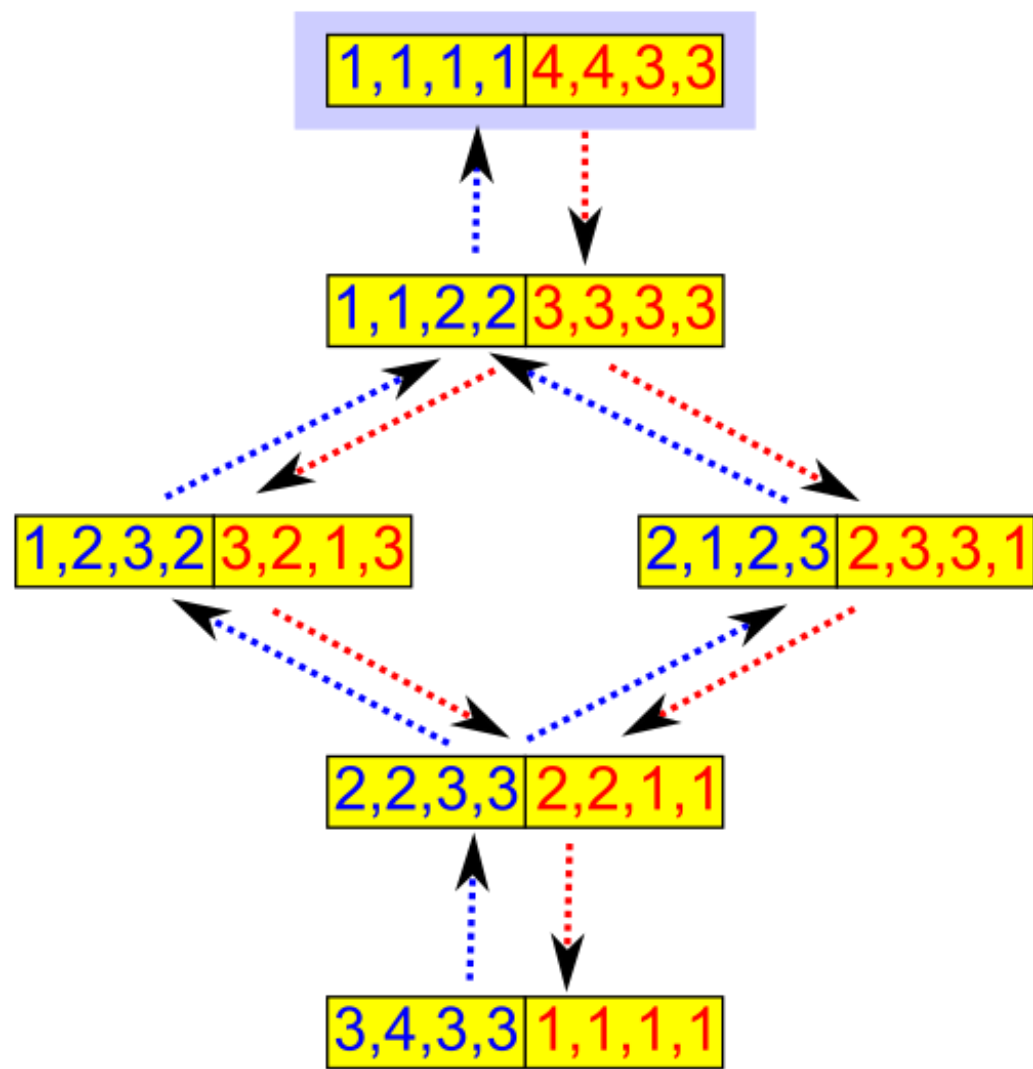


Consider a uniform combination of all integral stable matchings.



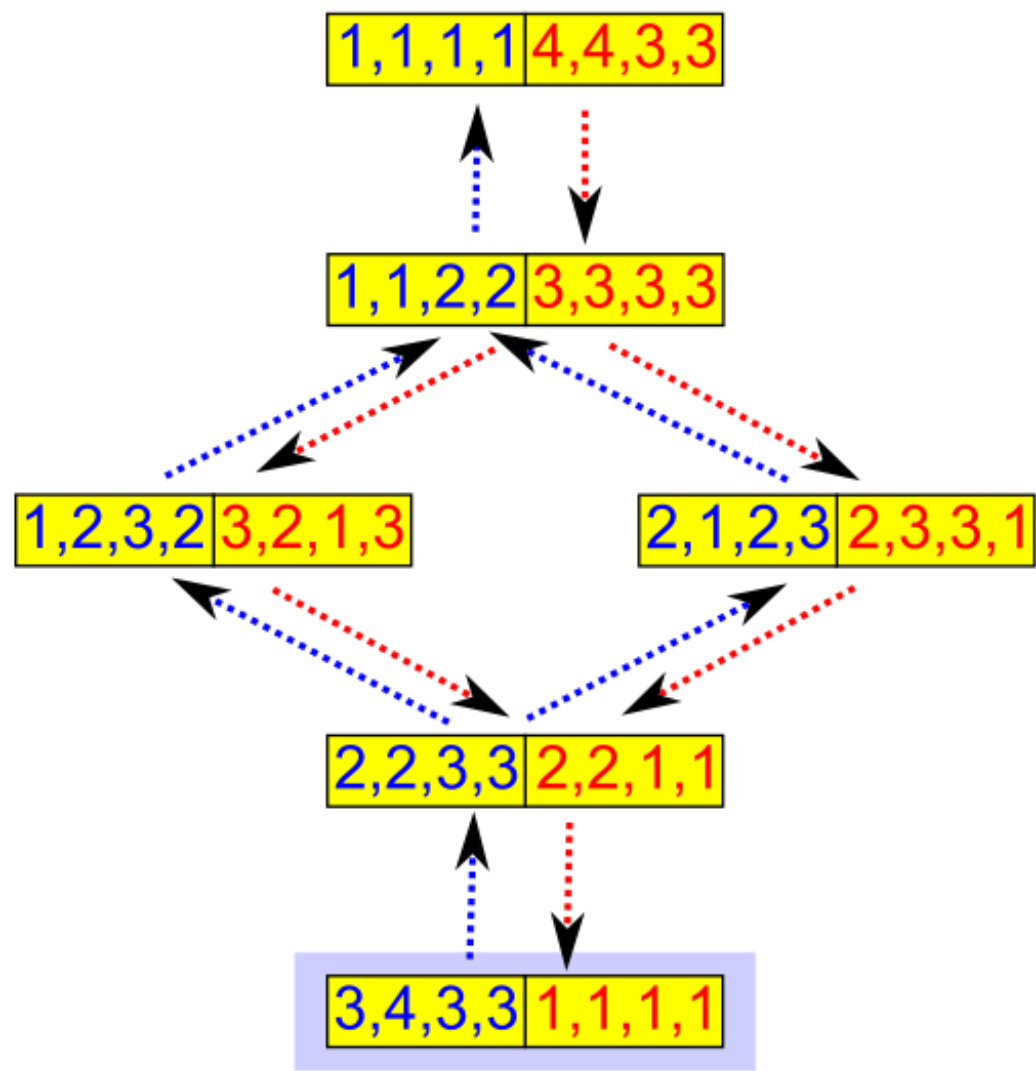
A vertical cut at any point in $[0,1]$ induces a stable matching.

Consider a uniform combination of all integral stable matchings.



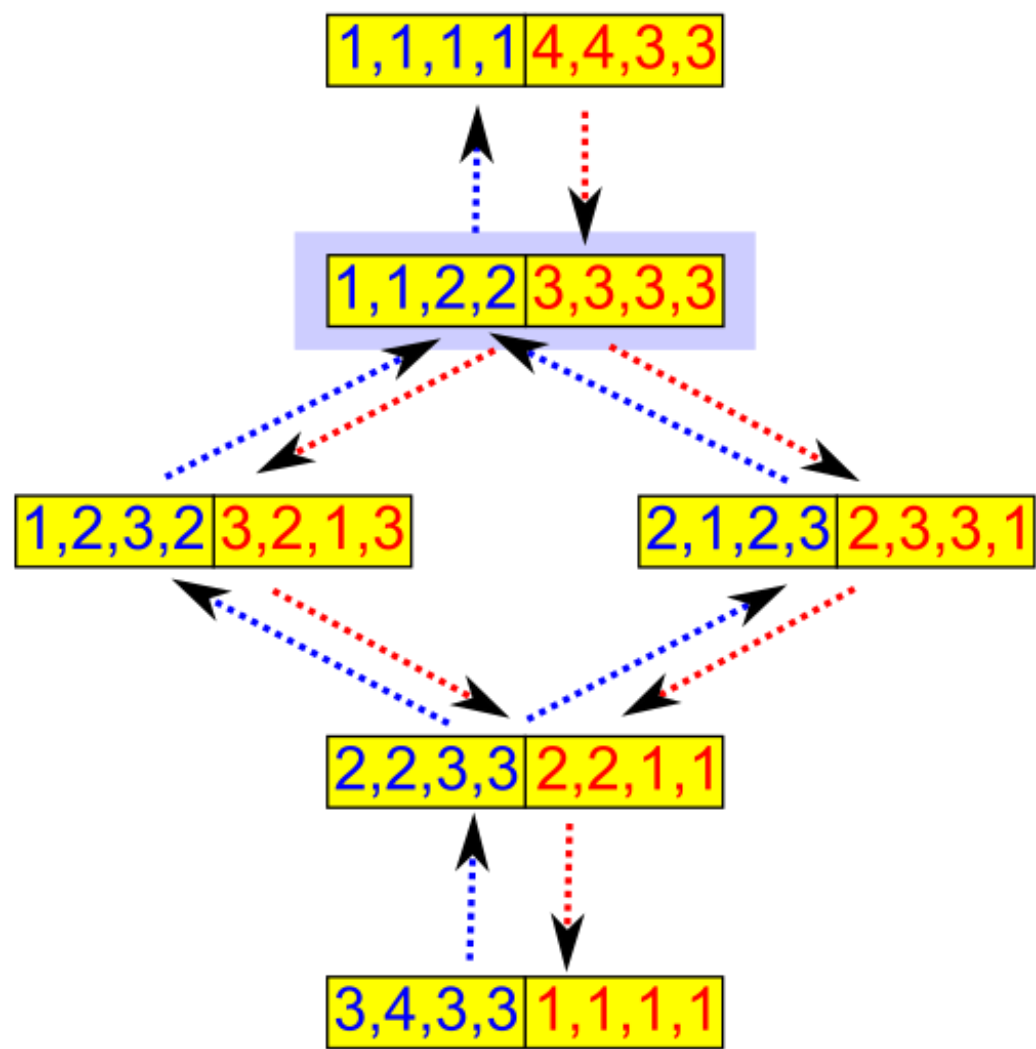
	0				1	
m_1		1		2	3	
m_2		1		2	4	
m_3	1		2		3	
m_4	1		2		3	
w_1	4		3		2	1
w_2	4		3		2	1
w_3		3			1	
w_4		3			1	

Consider a uniform combination of all integral stable matchings.



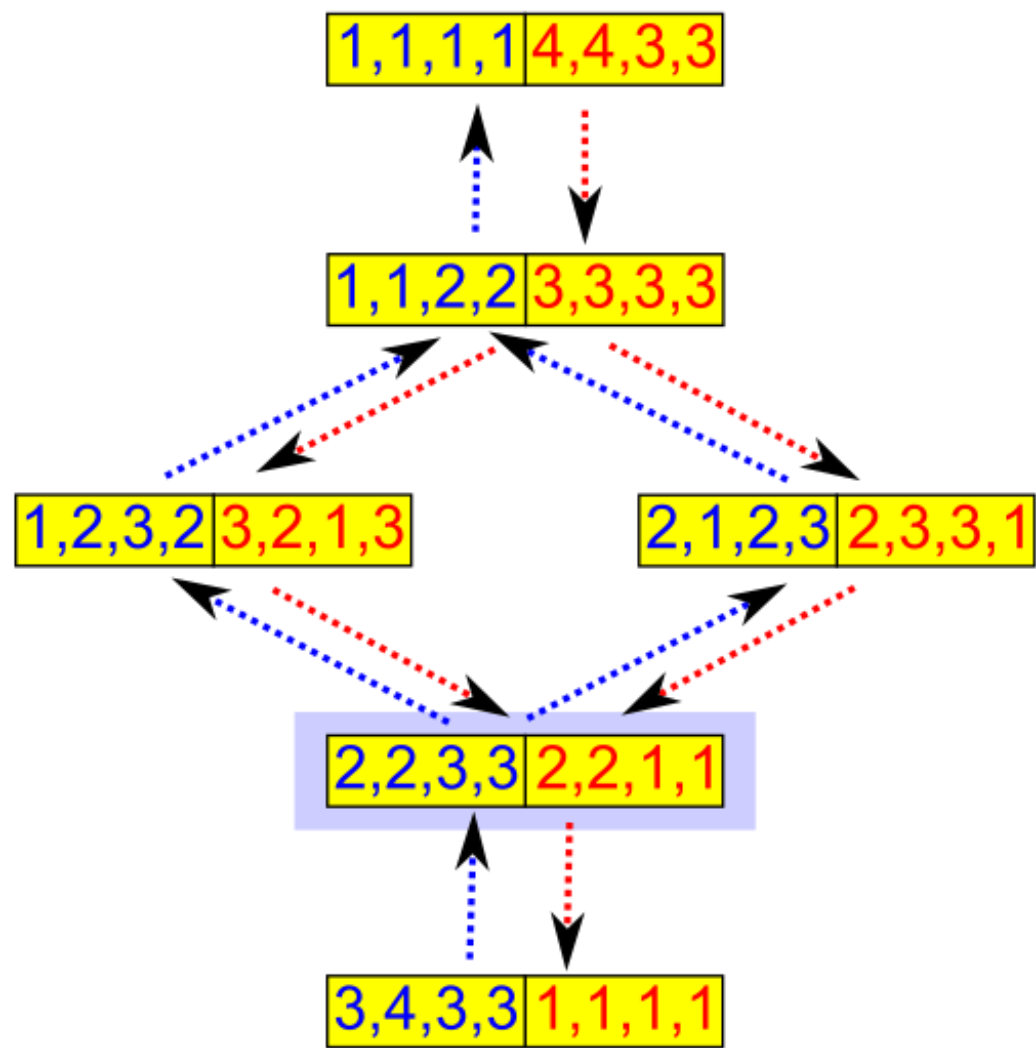
	0				1
m_1		1		2	3
m_2		1		2	4
m_3	1		2		3
m_4	1		2		3
w_1	4		3		2
w_2	4		3		2
w_3		3			1
w_4		3			1

Consider a uniform combination of all integral stable matchings.



	0		1
m_1		1	2 3
m_2		1	2 4
m_3	1	2	3
m_4	1	2	3
w_1	4	3	2 1
w_2	4	3	2 1
w_3		3	1
w_4		3	1

Consider a uniform combination of all integral stable matchings.



	0		1
m_1	1	2	3
m_2	1	2	4
m_3	1	2	3
m_4	1	2	3
w_1	4	3	2
w_2	4	3	2
w_3	3	1	1
w_4	3	1	1

Note: A vertical blue shaded region covers the column corresponding to the value '2' in the table.

Bad News

Bad News

[Cheng, *Algorithmica* 2010]

Computing a median stable matching is **#P-hard**.

Bad News

[Cheng, *Algorithmica* 2010]

Computing a median stable matching is **#P-hard**.

as hard as...

- computing the permanent of a 0-1 matrix
- counting the number of perfect matchings in a bipartite graph
- counting the number of stable matchings for a given instance
- and many others...

Other Notions of Fairness

Other Notions of Fairness

Egalitarian

[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the
average rank of matched partners
across all men and women

Other Notions of Fairness

Egalitarian

[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the *average rank* of matched partners across all men and women

Minimum regret

[Knuth, 1976 (attributed to Selkow)]

a stable matching that minimizes the *maximum rank* of matched partner of any man or woman

Other Notions of Fairness

Egalitarian

[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the *average rank* of matched partners across all men and women

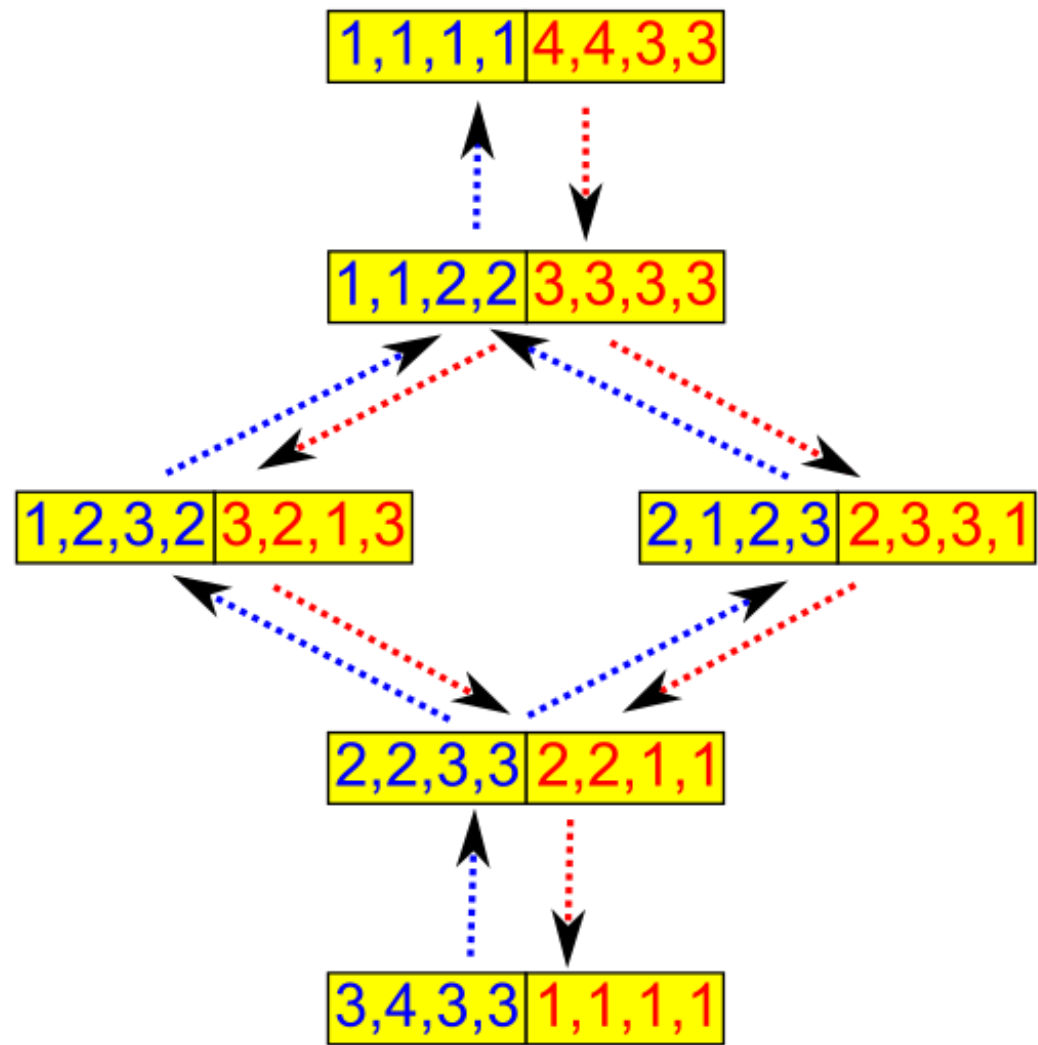
Polynomial time

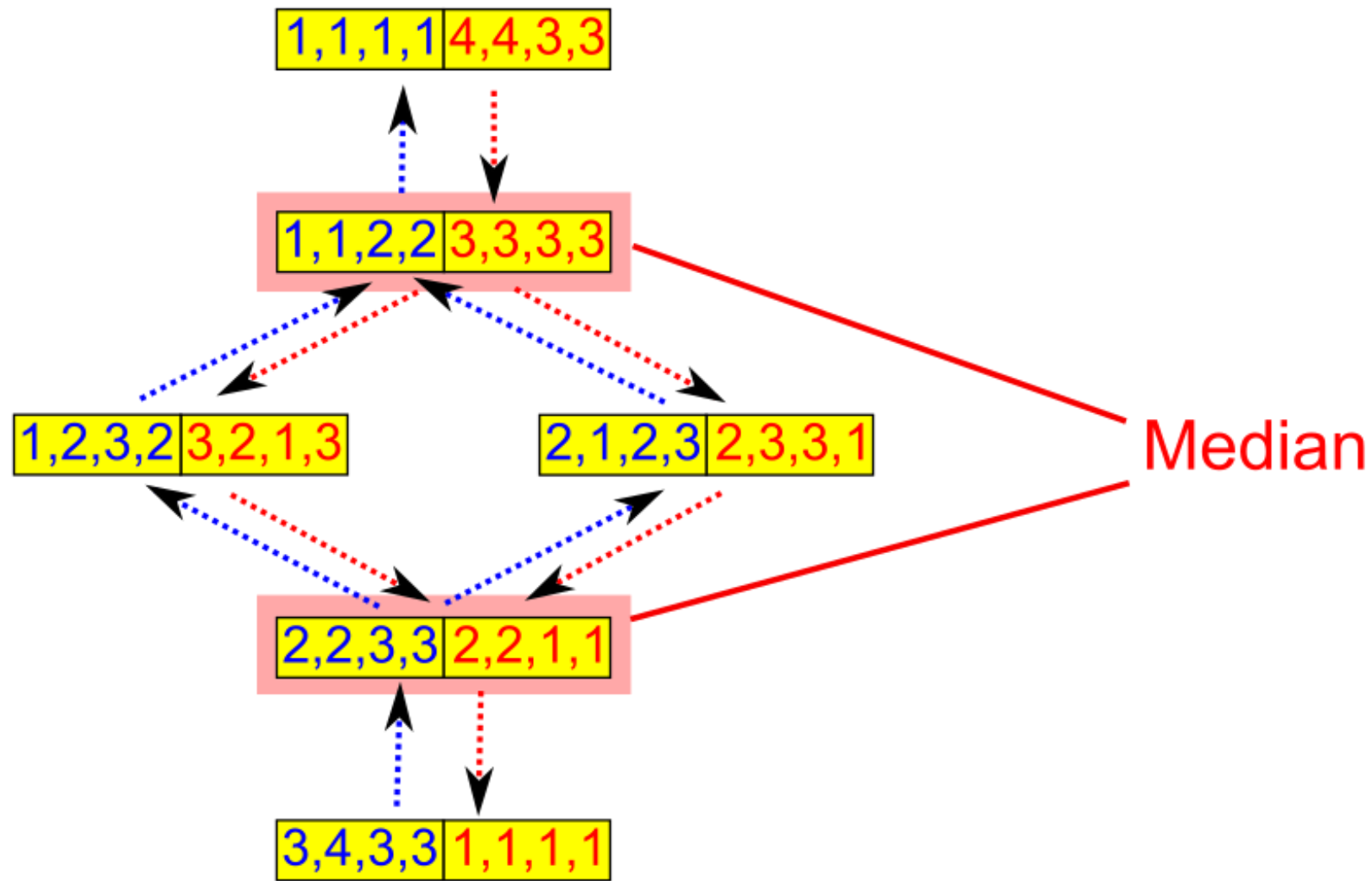
Minimum regret

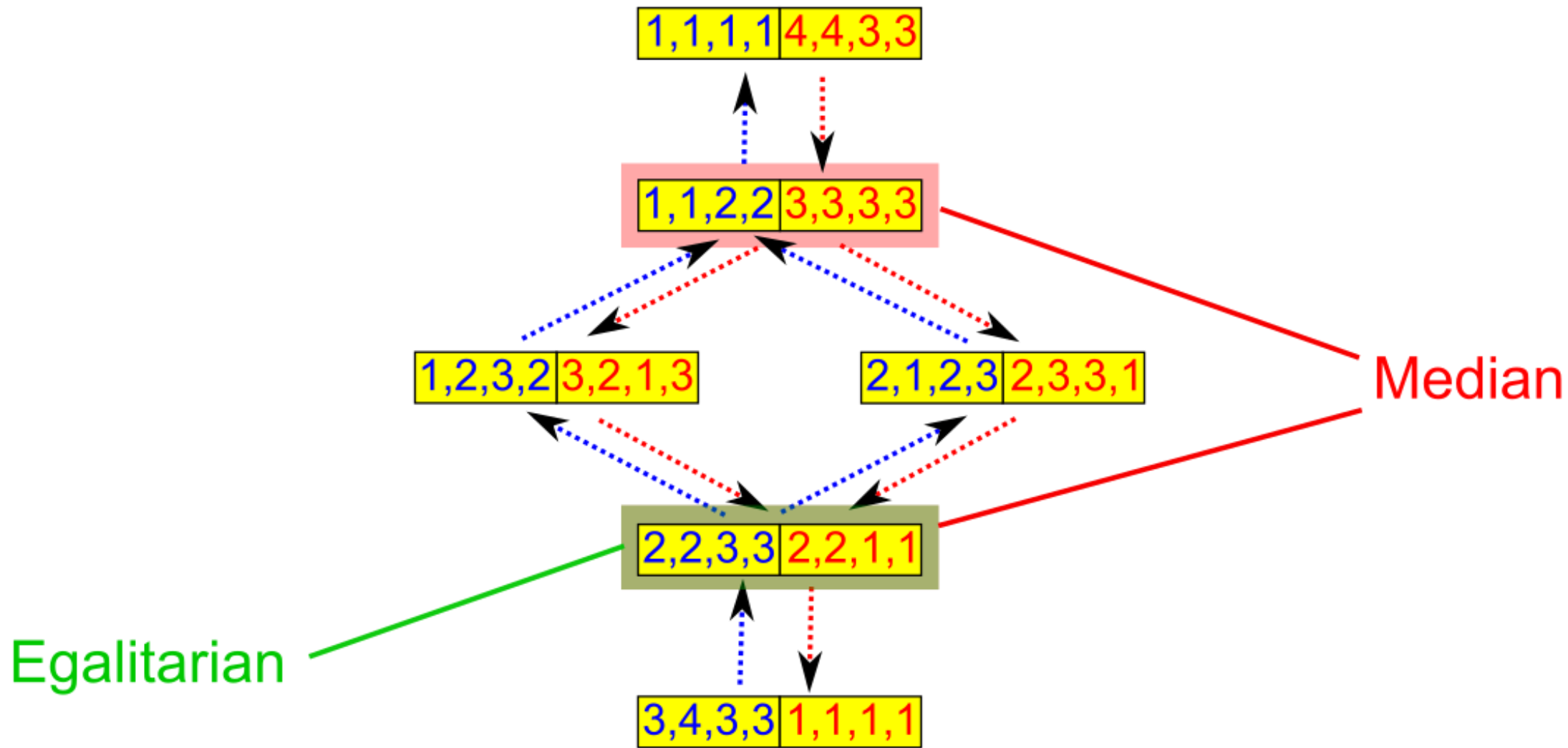
[Knuth, 1976 (attributed to Selkow)]

a stable matching that minimizes the *maximum rank* of matched partner of any man or woman

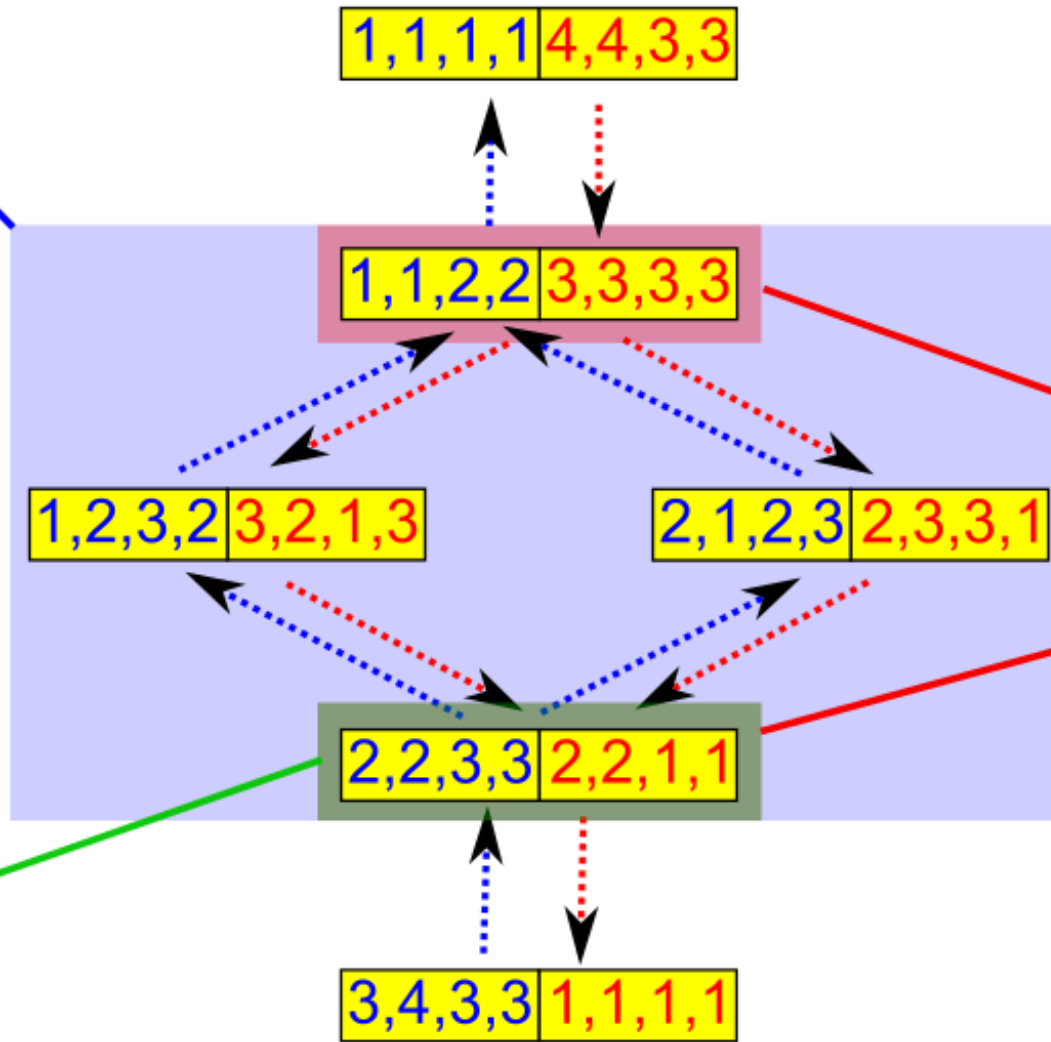
Polynomial time







Minimum Regret

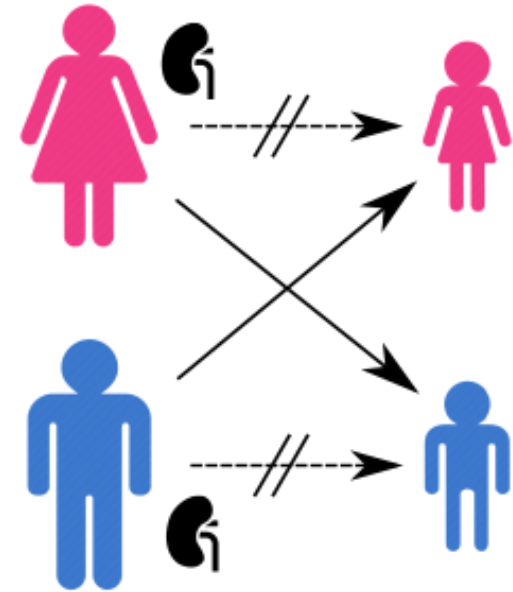
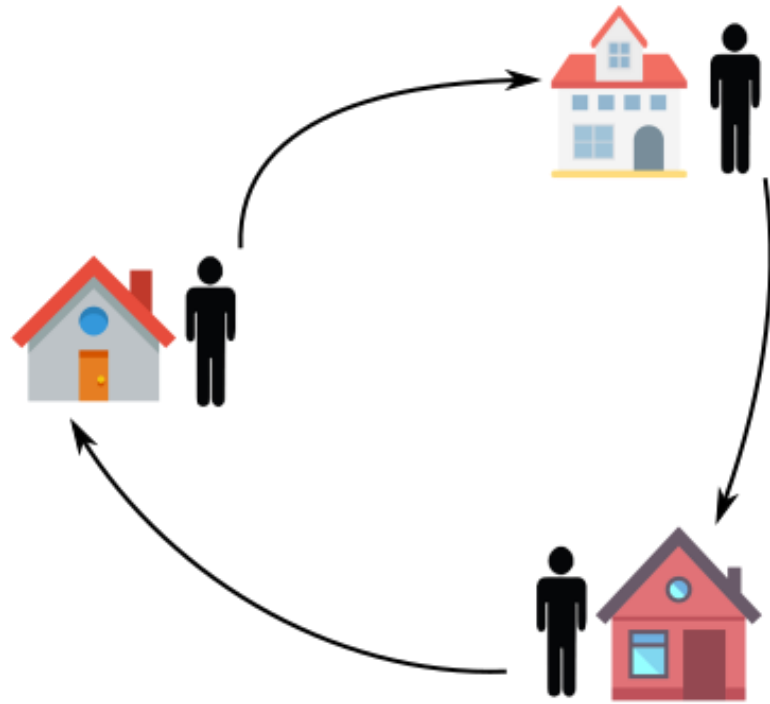


Median

Egalitarian

Next Time

House allocation and kidney exchange



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<https://www.optimalmatching.com/MATCHUP2015/slides/ChristineCheng.pdf>

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