

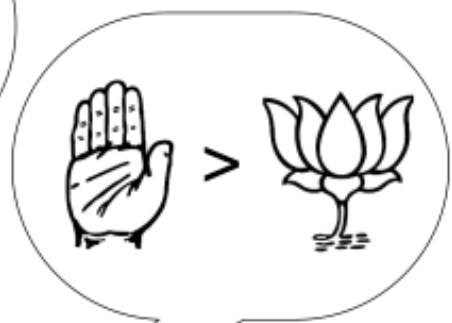
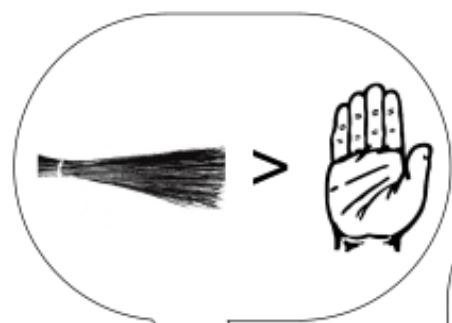
# Lecture 1

## Stable Matching Problem

# What this course is about

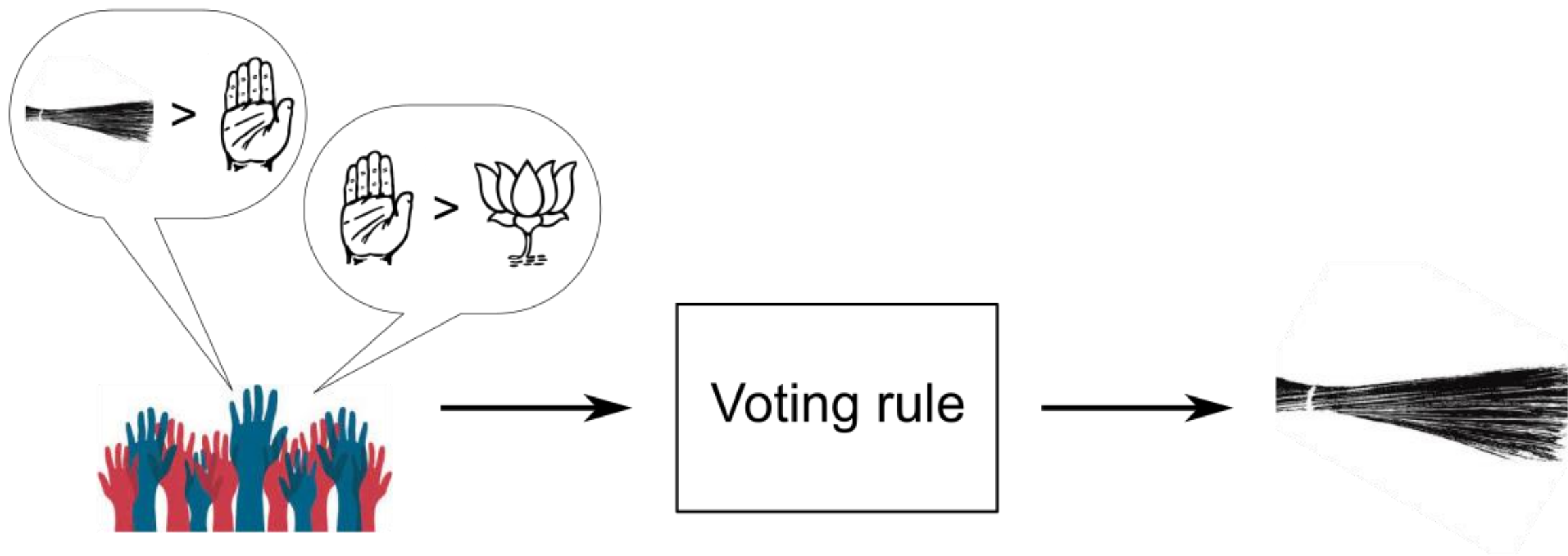


Understanding the role of **computation**  
in **collective decision-making** problems



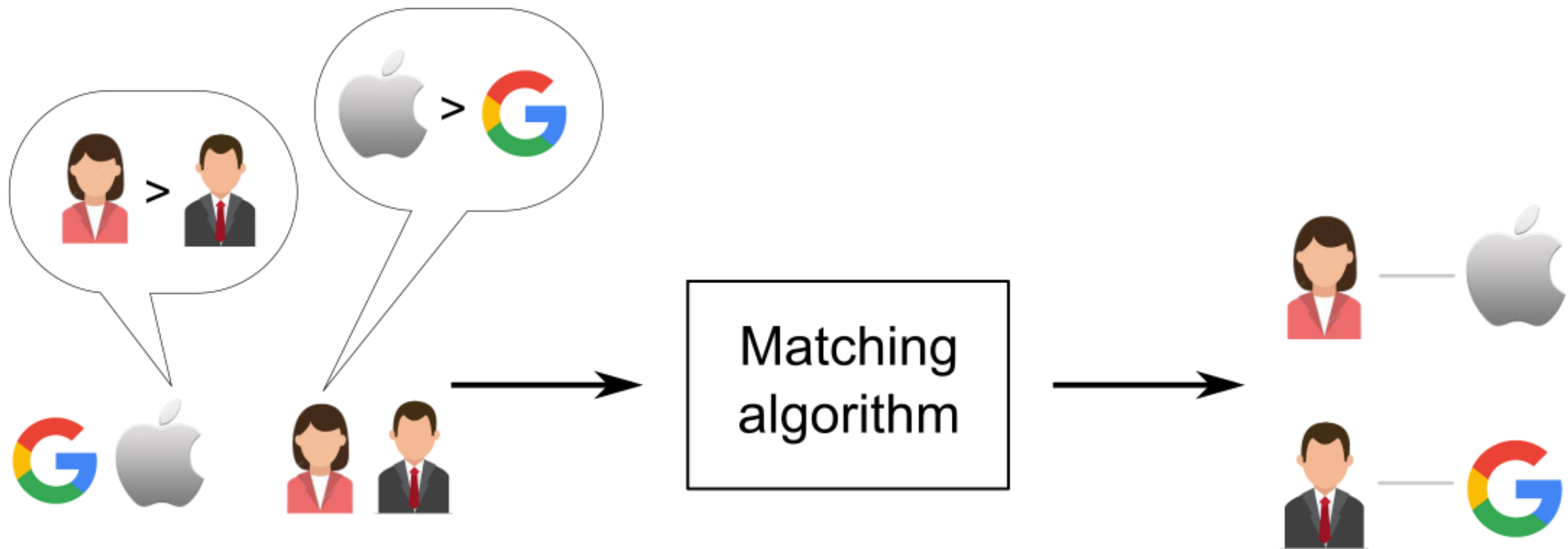
Voting rule

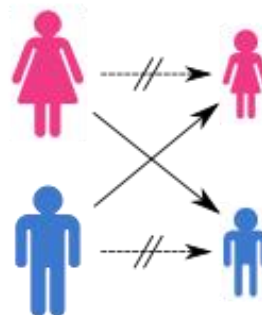
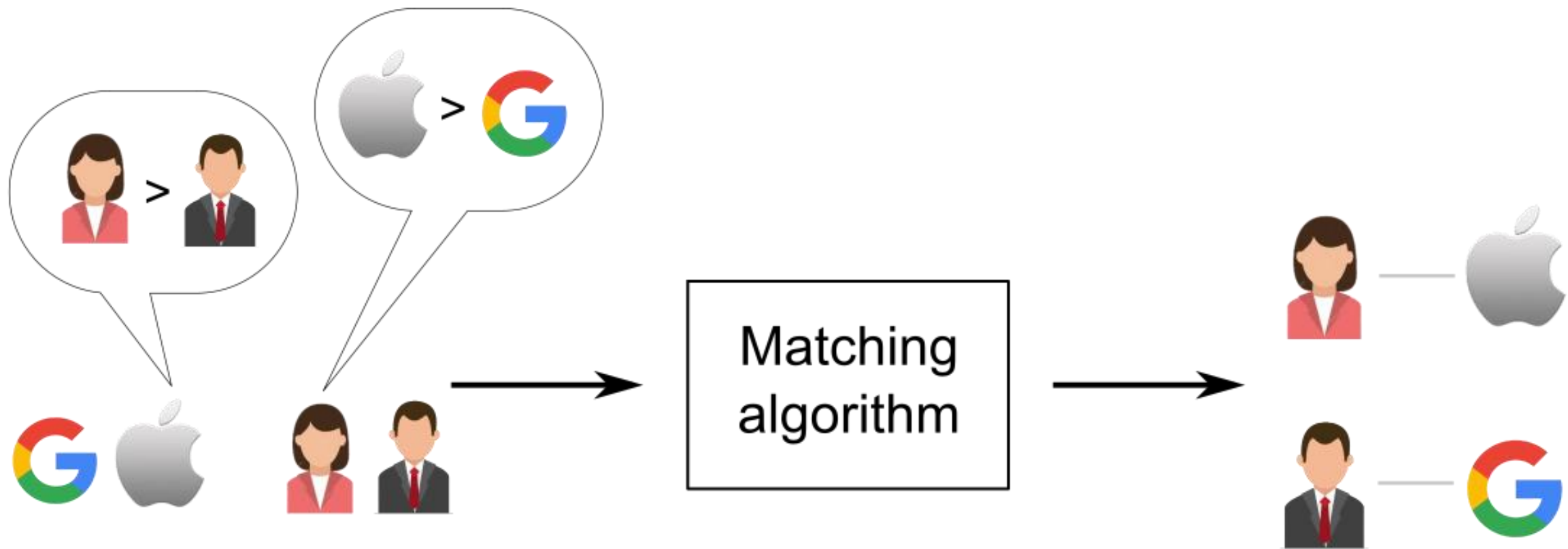




amazon









Allocation  
mechanism





Allocation  
mechanism



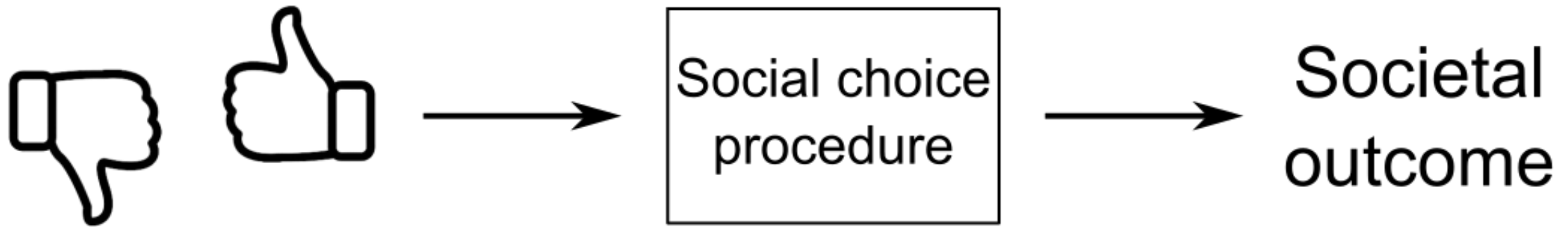


# Social Choice

Making a **collective** decision from **individual** preferences

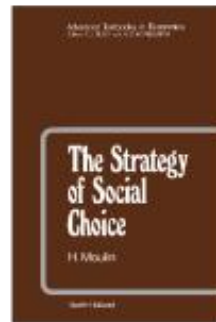
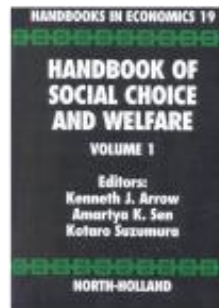
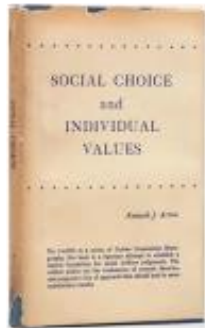
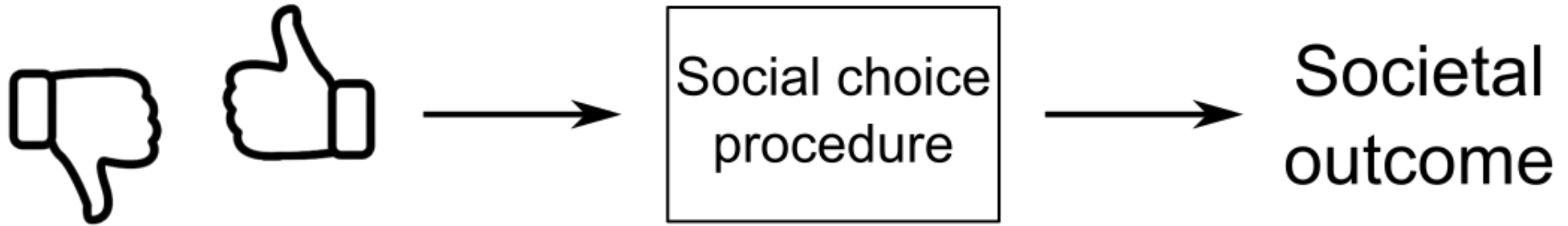
# Social Choice

Making a **collective** decision from **individual** preferences



# Social Choice

Making a **collective** decision from **individual** preferences



Arrow



Sen



Maskin



Roth



Shapley

## Classical Social Choice

Does there exist a social choice procedure  
with the desired economic properties?

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Does there **exist** a social choice procedure with the desired economic properties?



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## Classical Social Choice

Does there **exist** a social choice procedure with the desired economic properties?

## Computational Social Choice

(This course)

How does **computation** influence the economic properties of social choice procedures?





LEC 1-5





LEC 1-5



LEC 6-11



LEC 13-17



LEC 1-5



LEC 6-11



LEC 13-17



LEC 1-5



LEC 6-11

LEC 18-23



LEC 13-17

LEC 1-5

LEC 6-11

LEC 18-23

LEC 12 and 24-26: Project Presentations



LEC 13-17

**LEC 1-5**

LEC 6-11

LEC 18-23

LEC 12 and 24-26: Project Presentations



# Stable Matching Problem

# Stable Matching Problem



# Stable Matching Problem

$w_1 > w_2 > w_3$



$m_3 > m_2 > m_1$

$w_2 > w_1 > w_3$



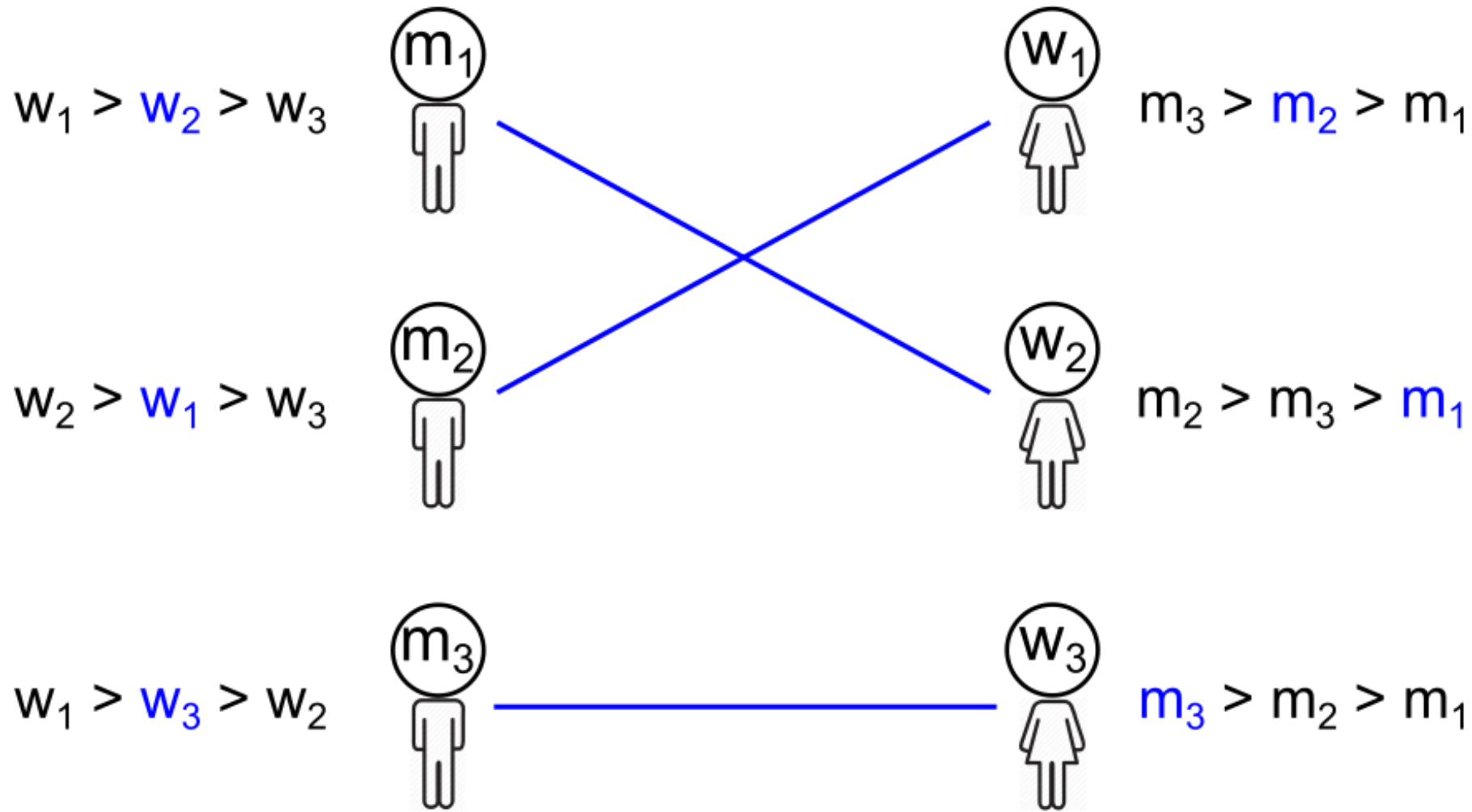
$m_2 > m_3 > m_1$

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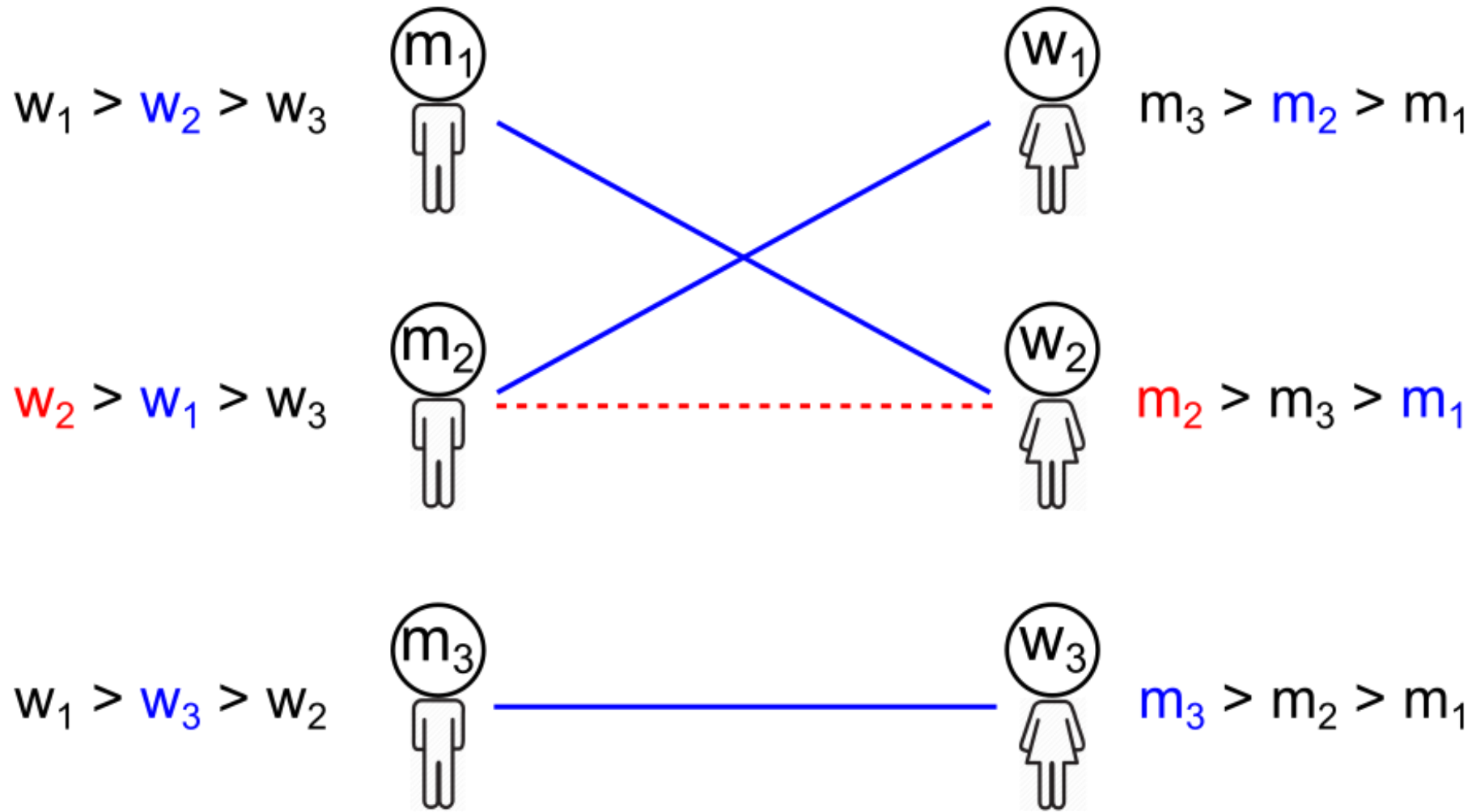


$m_3 > m_2 > m_1$

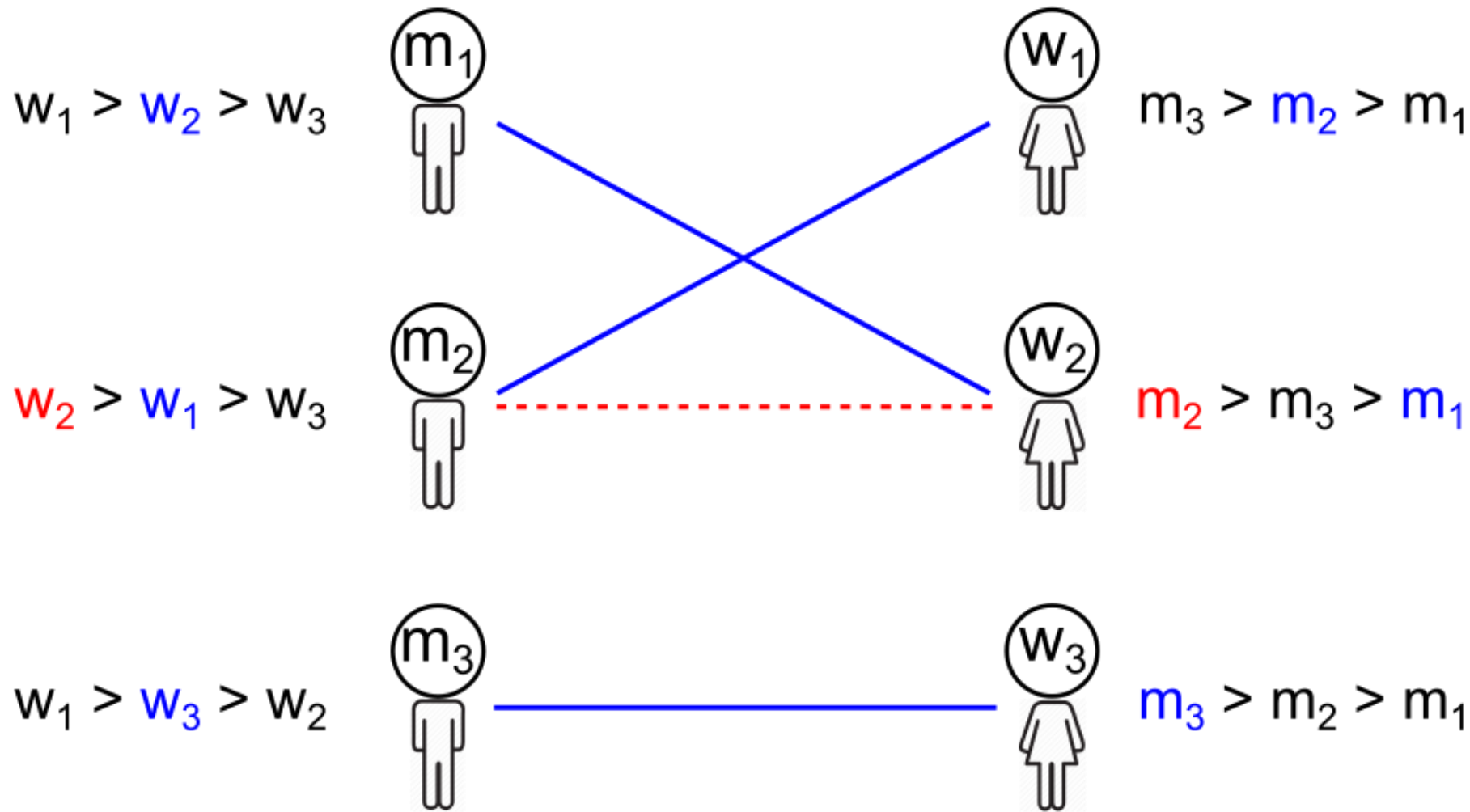
# Stable Matching Problem



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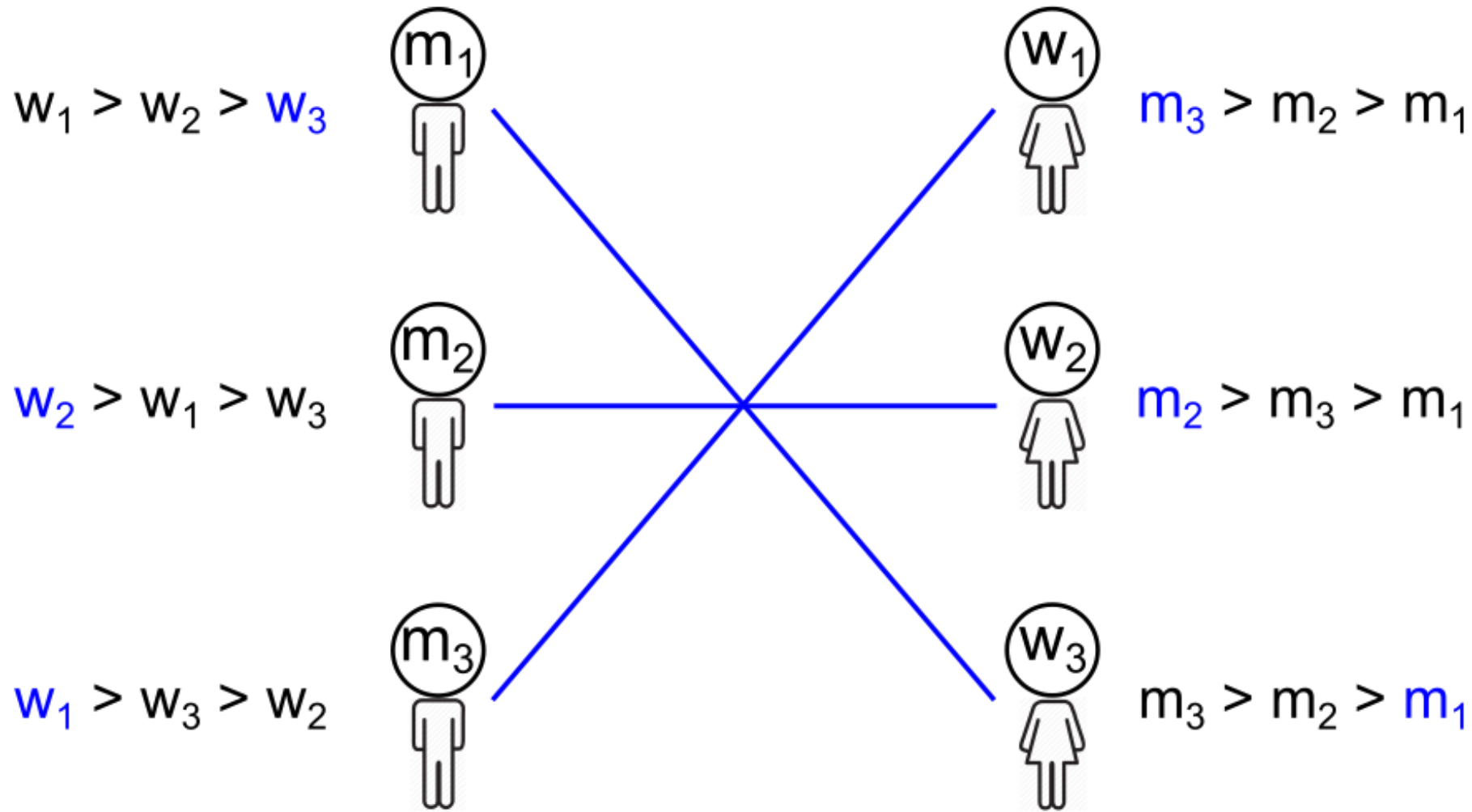


# Stable Matching Problem



A matching is **stable** if there is no **blocking pair**.

# Stable Matching Problem



A matching is **stable** if there is no **blocking pair**.



## COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE\* AND L. S. SHAPLEY, Brown University and the RAND Corporation



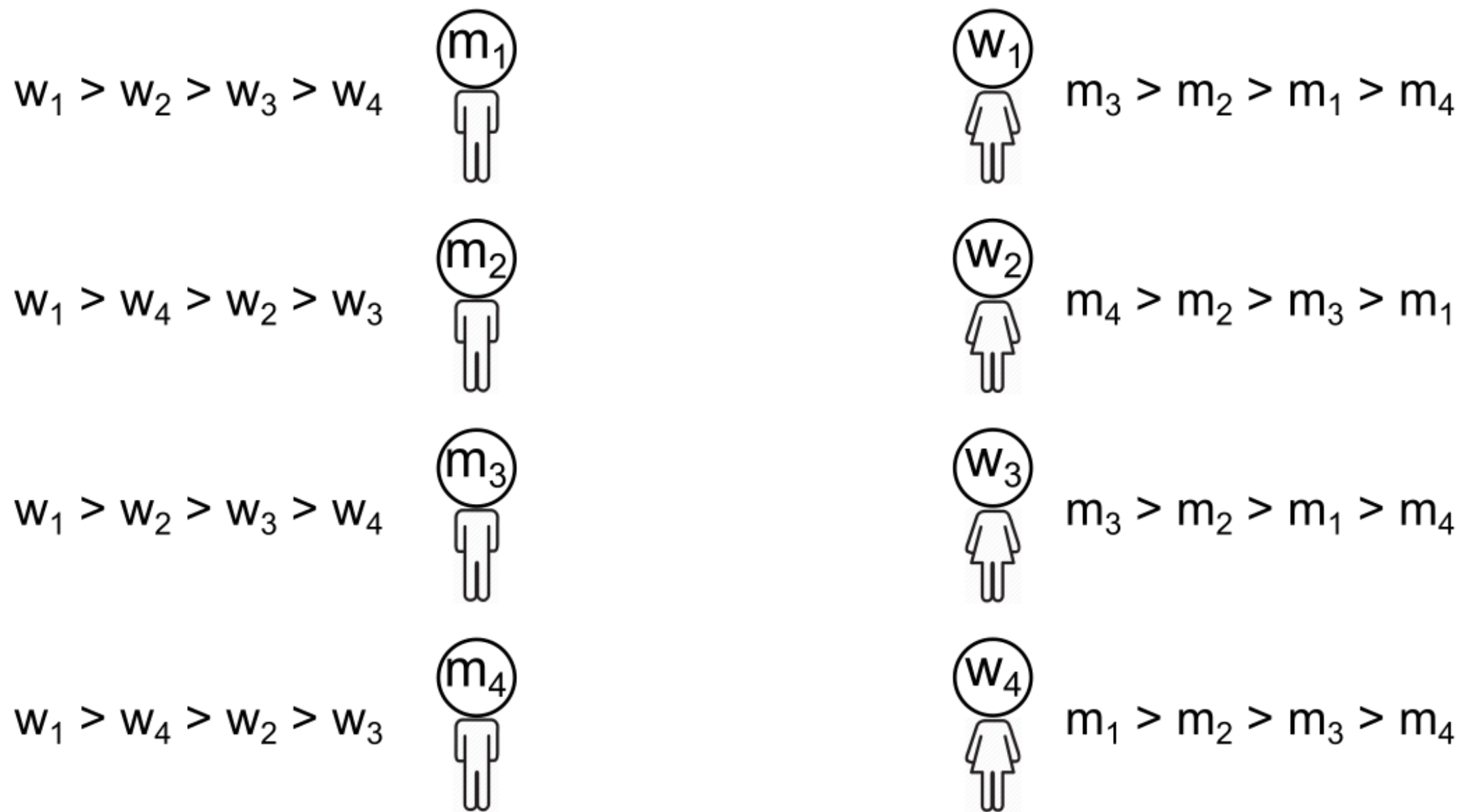
Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9-15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.



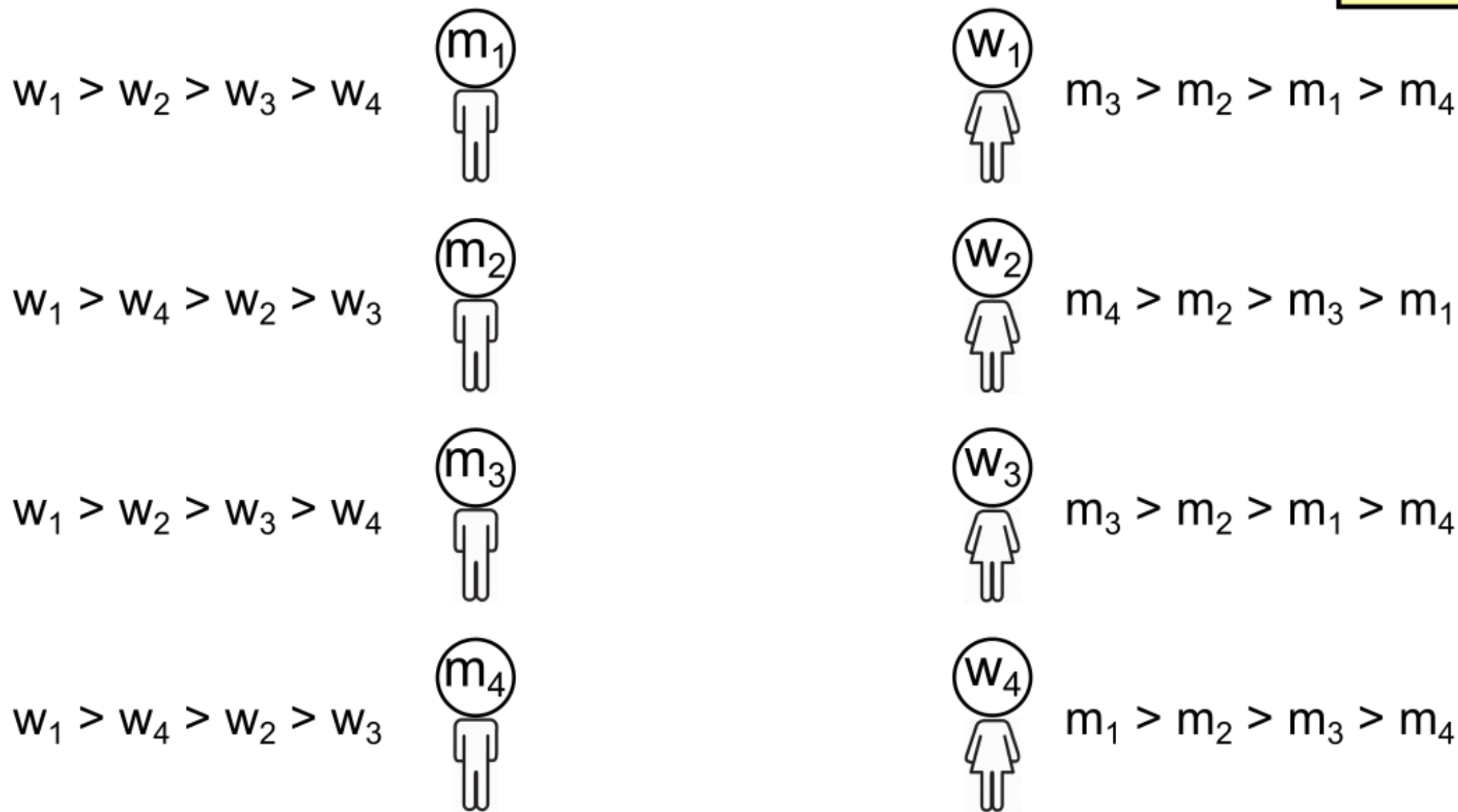
# Deferred-Acceptance Algorithm

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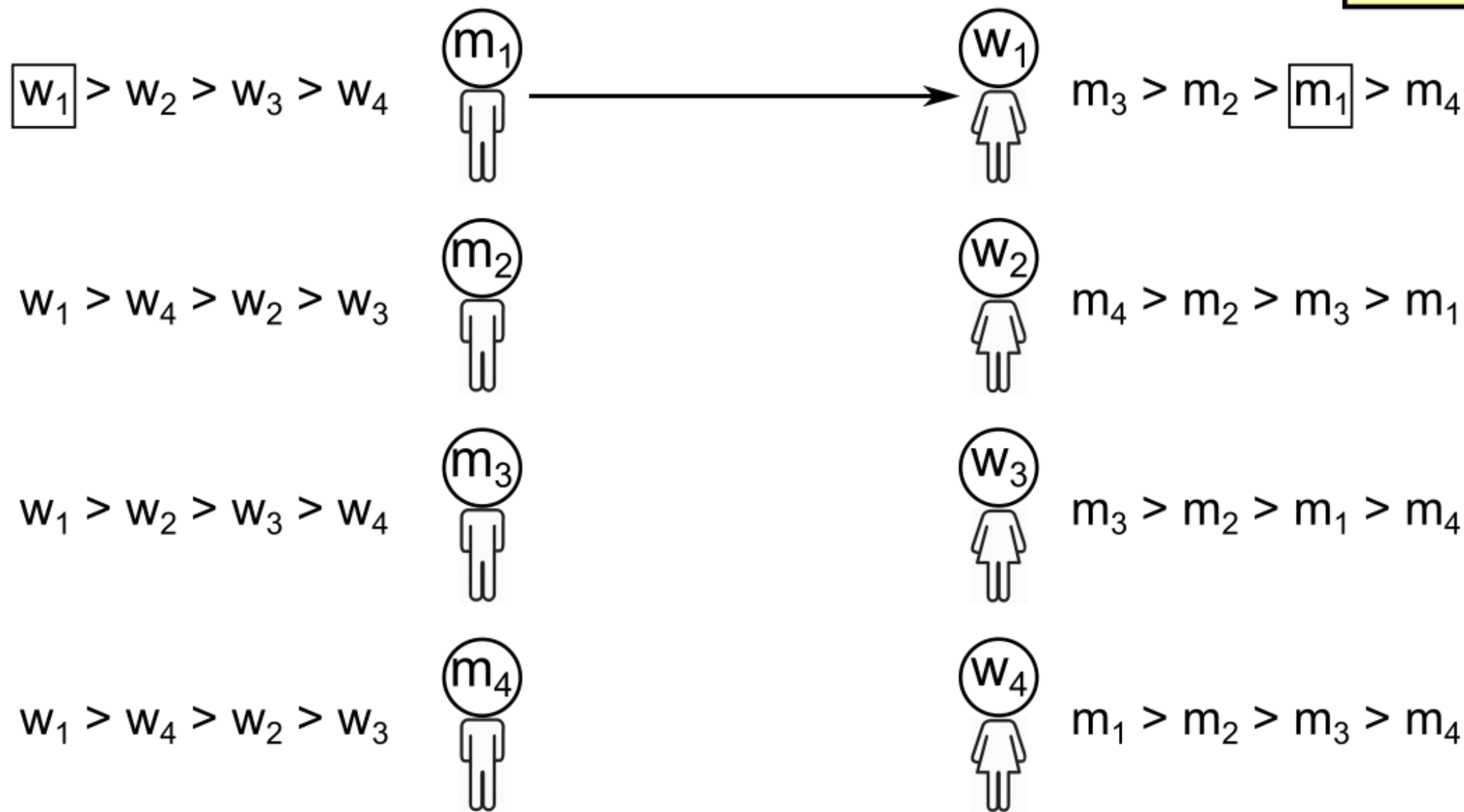
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Round 1



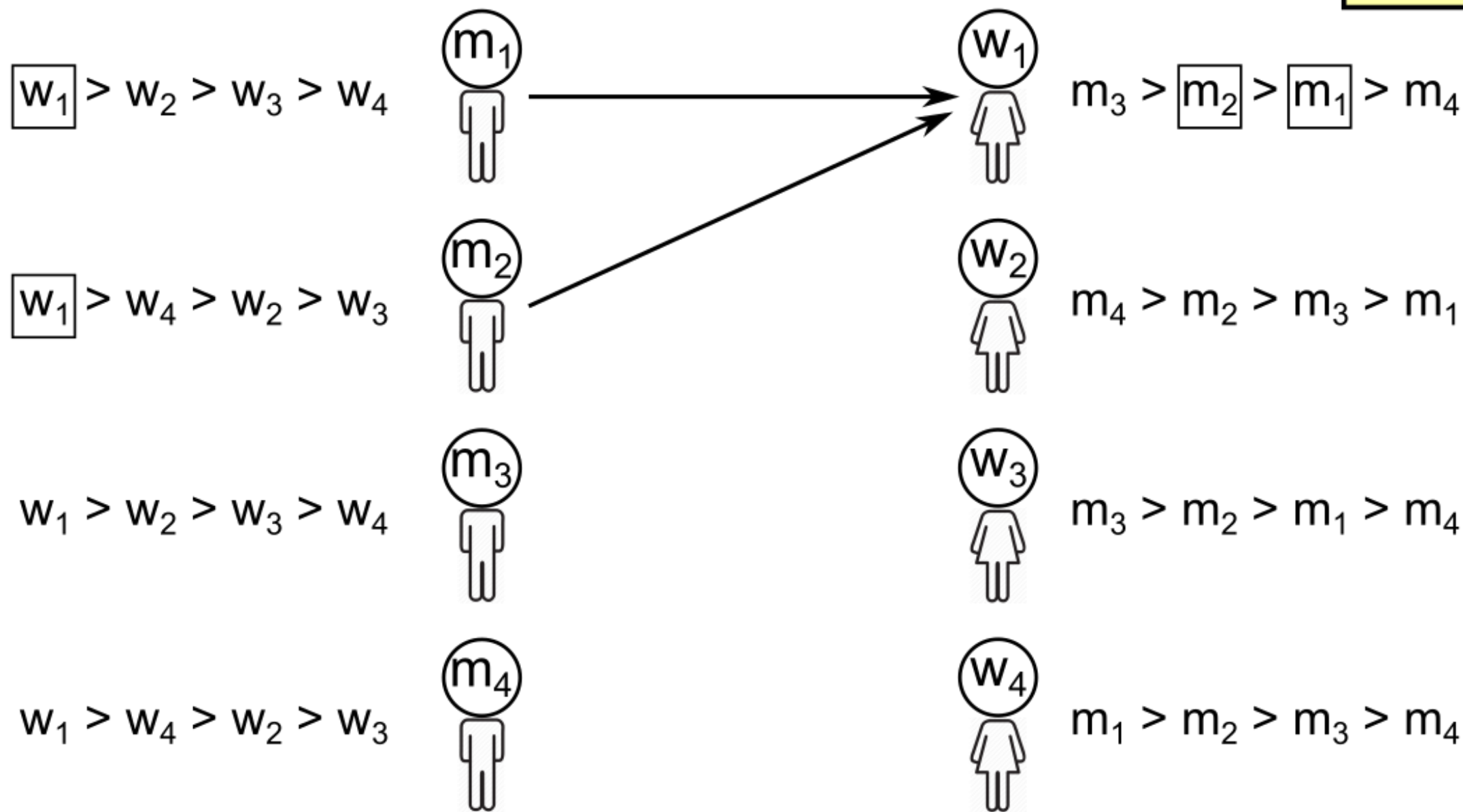
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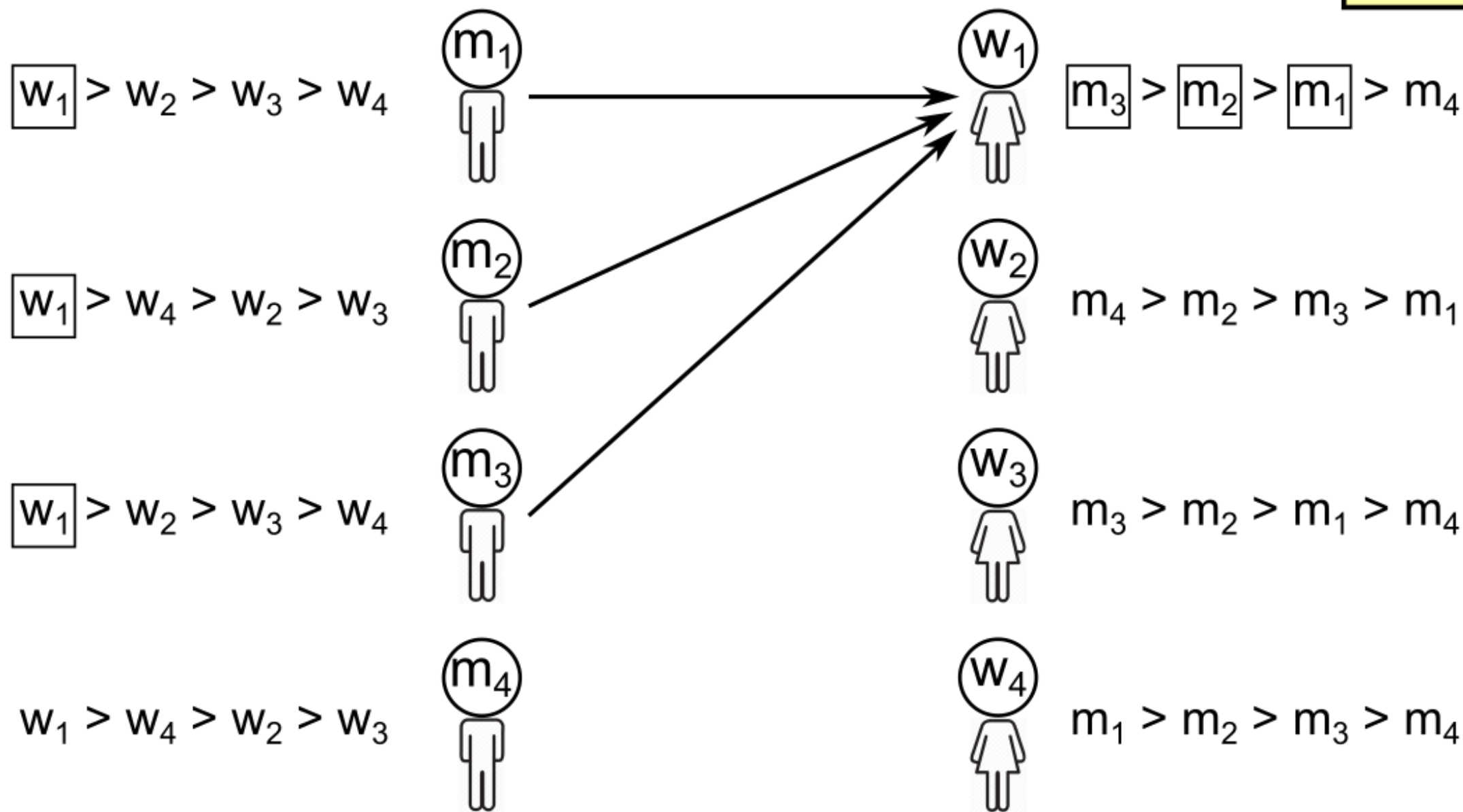
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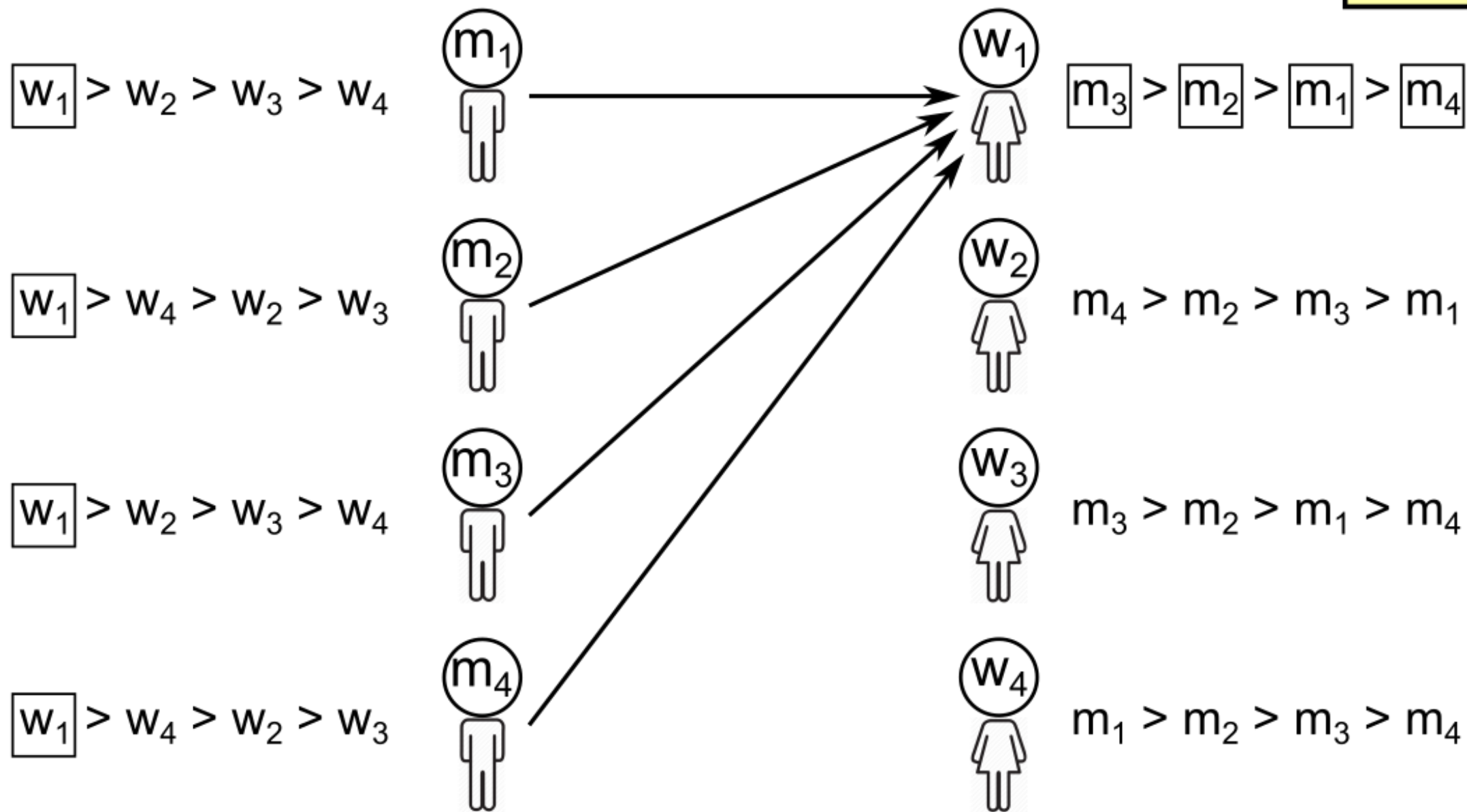
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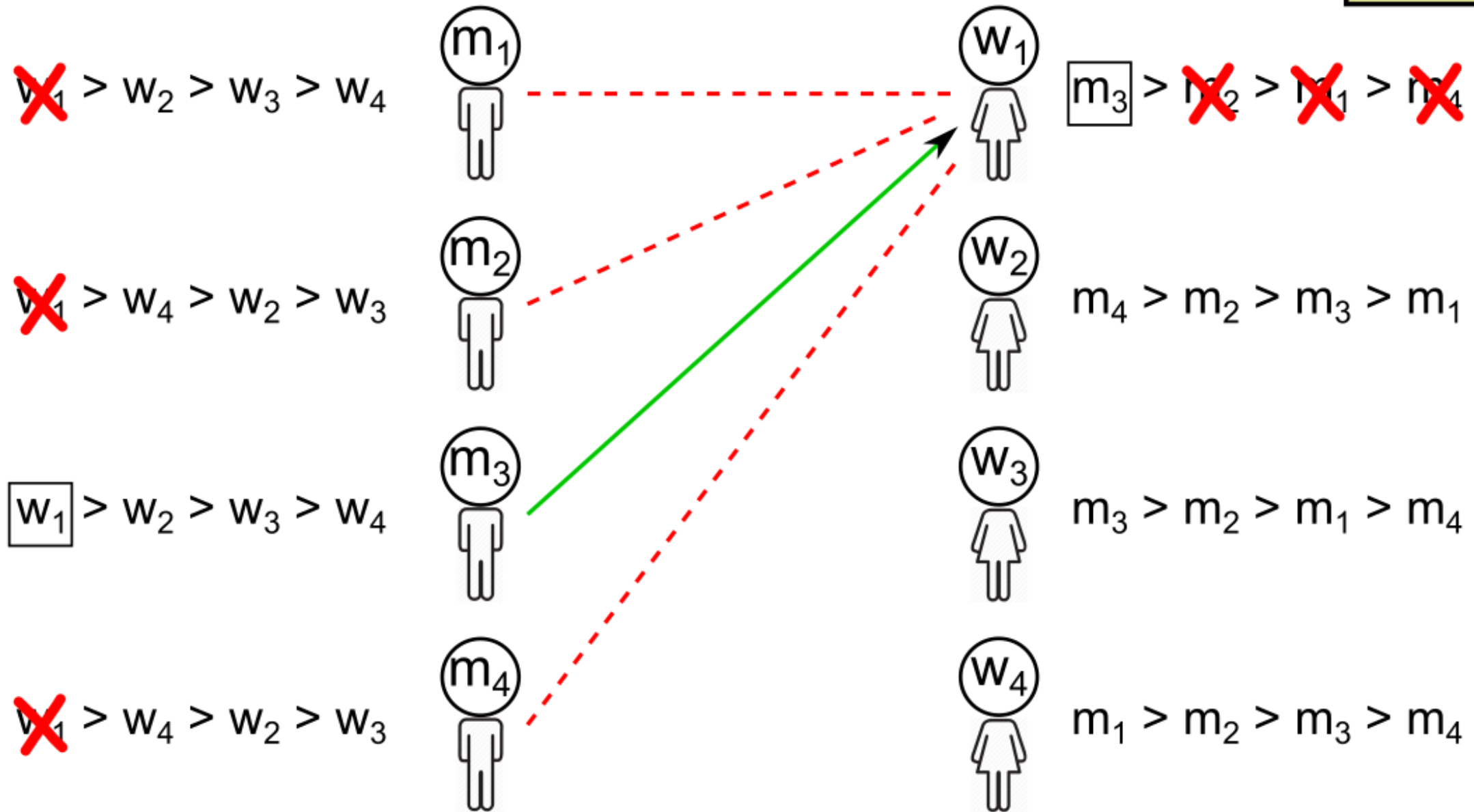
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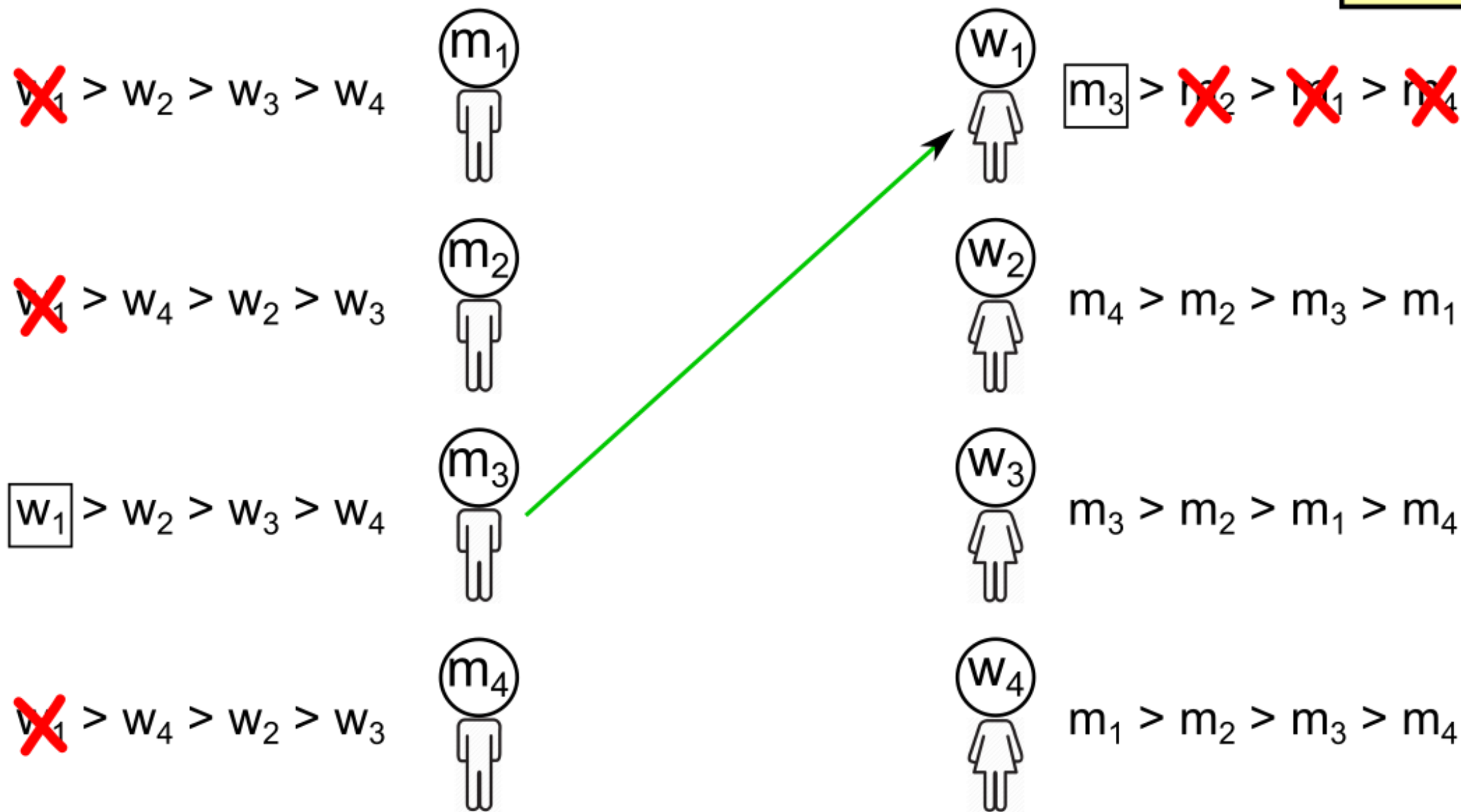
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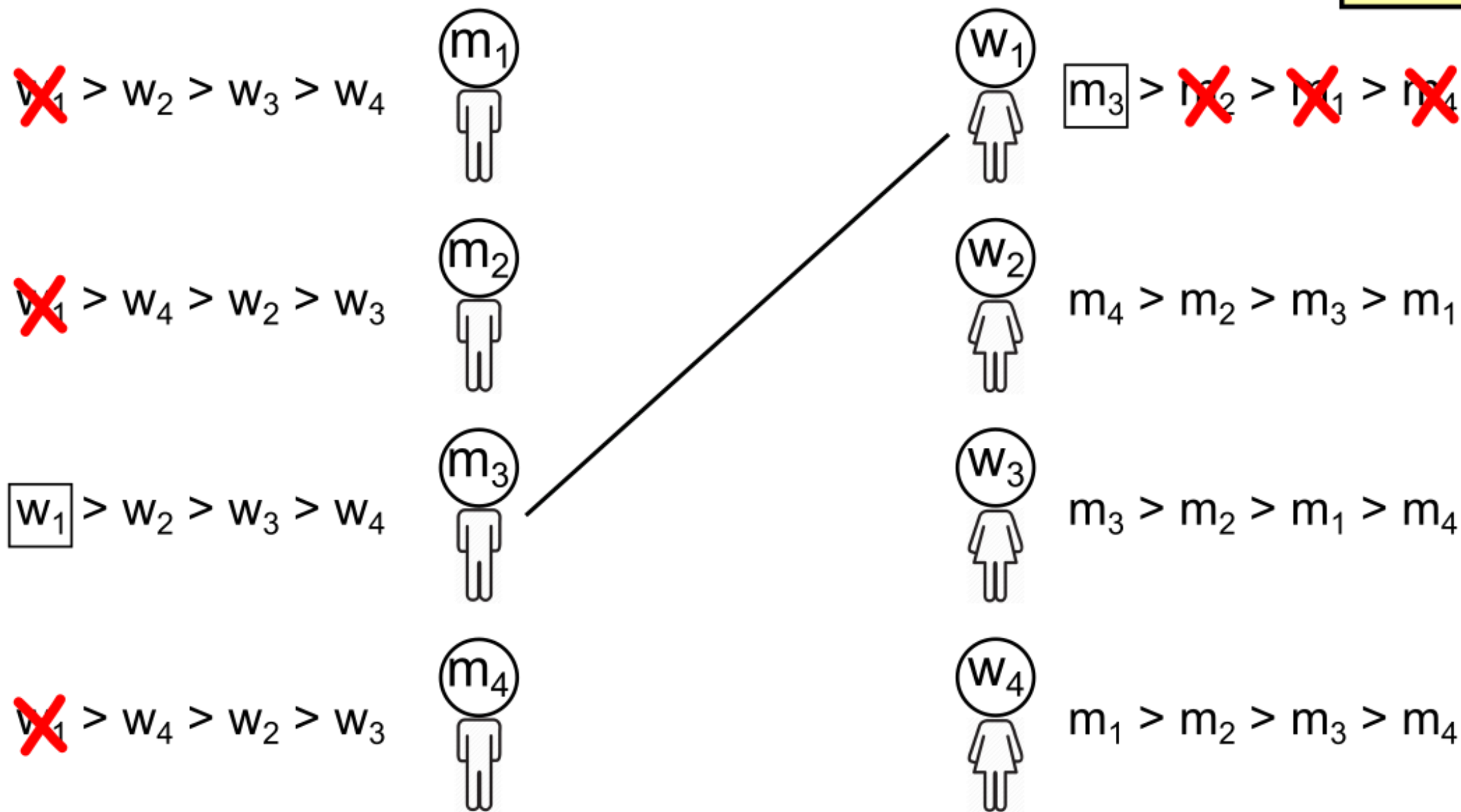
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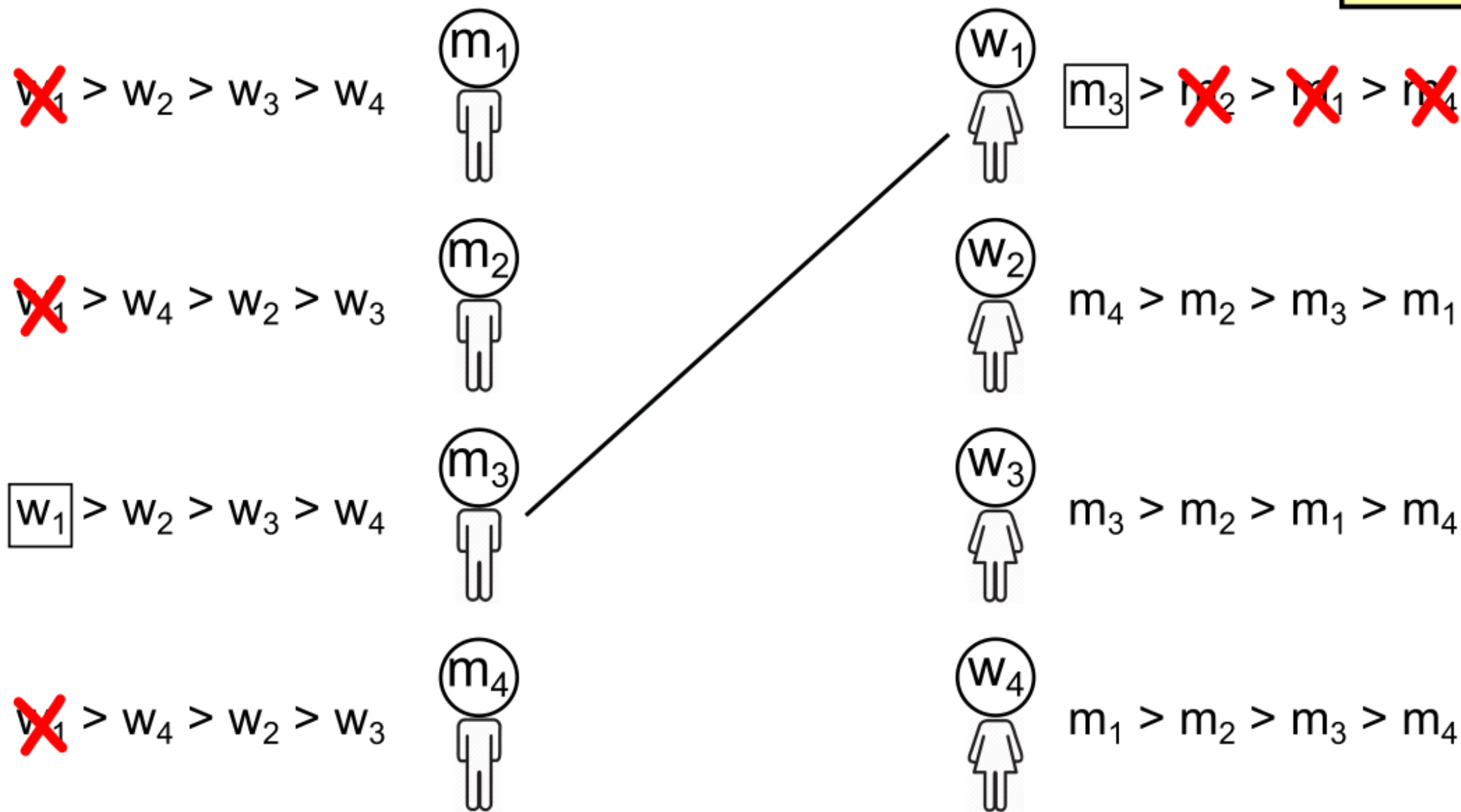
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Round 1



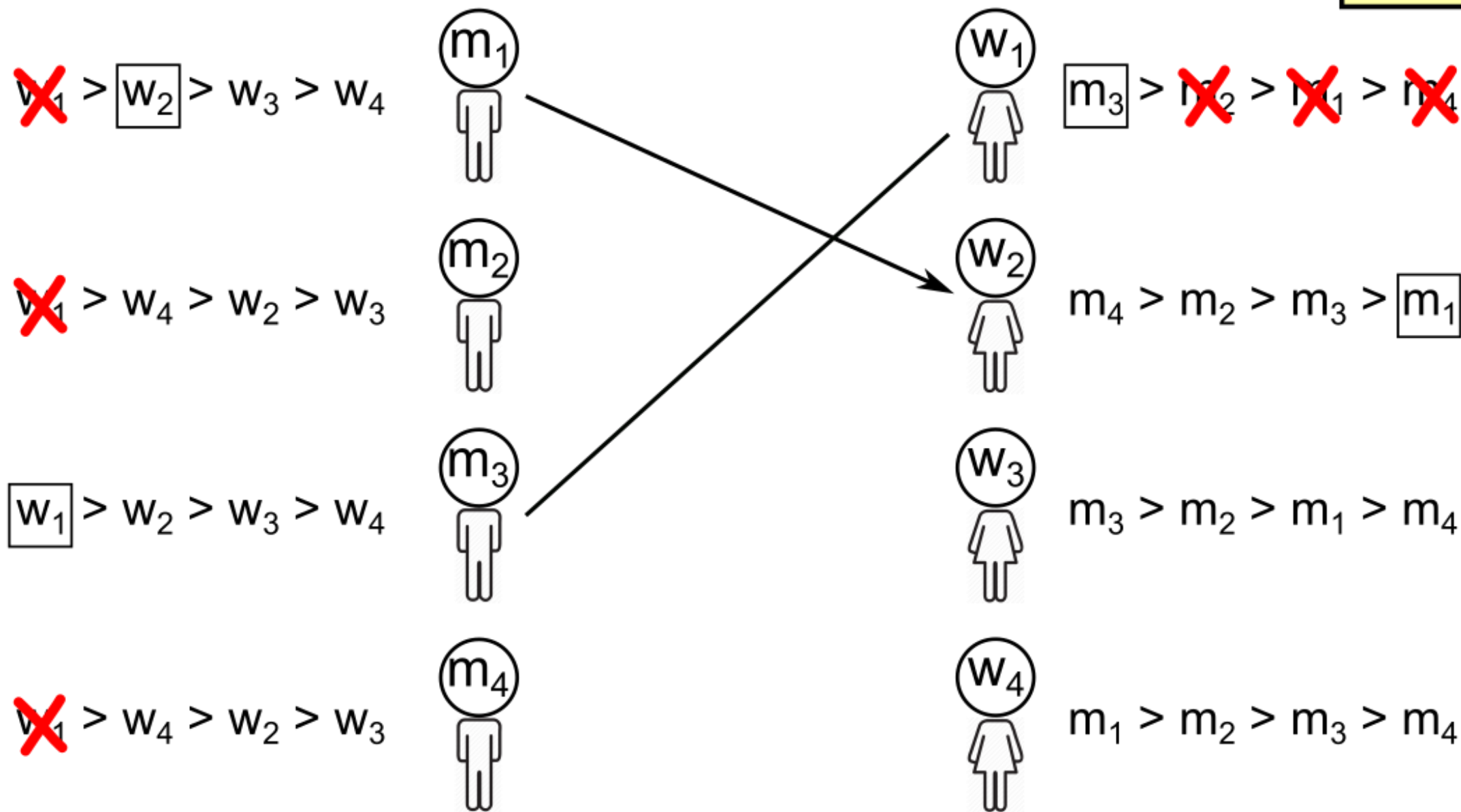
# Deferred-Acceptance Algorithm

Round 2



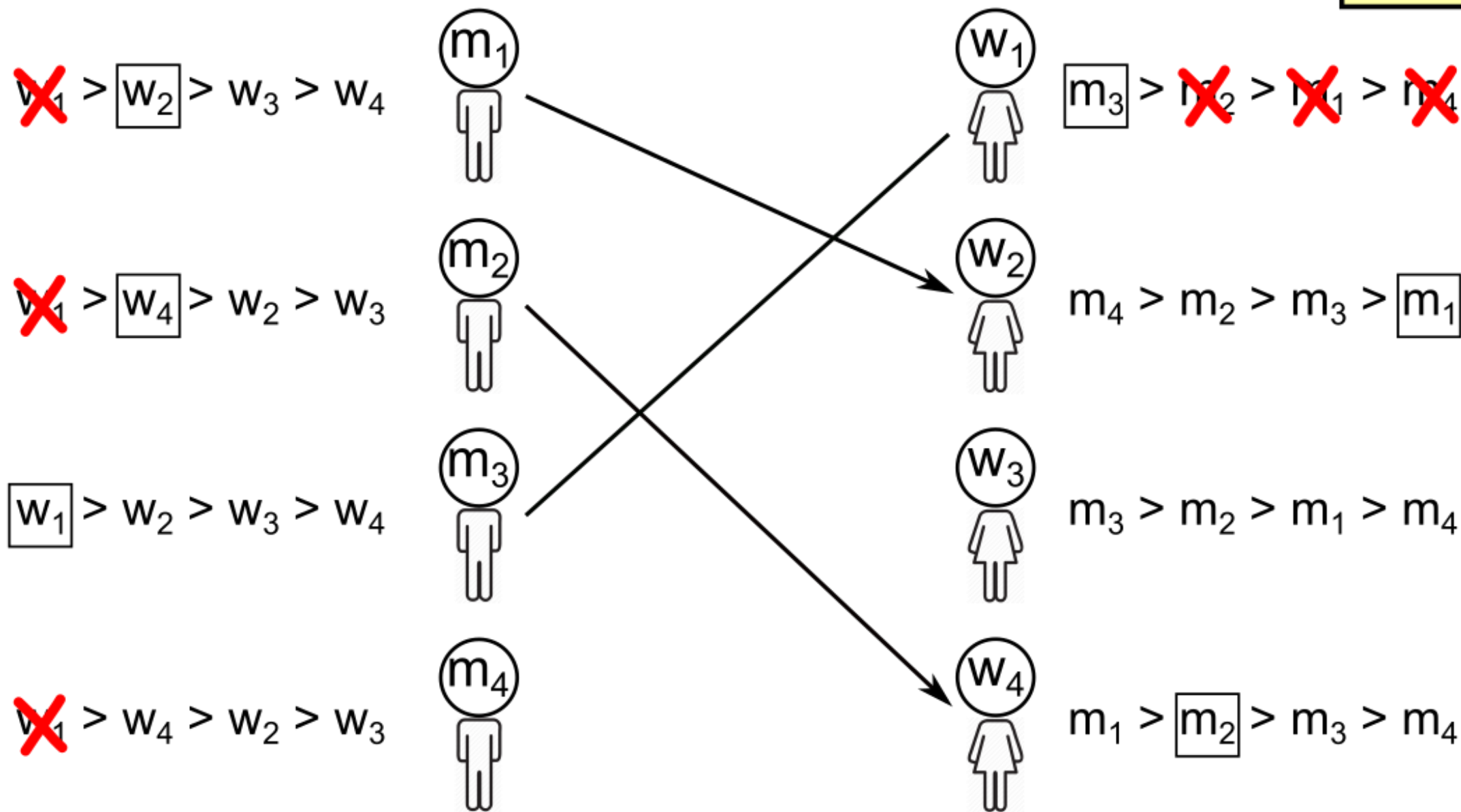
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Round 2



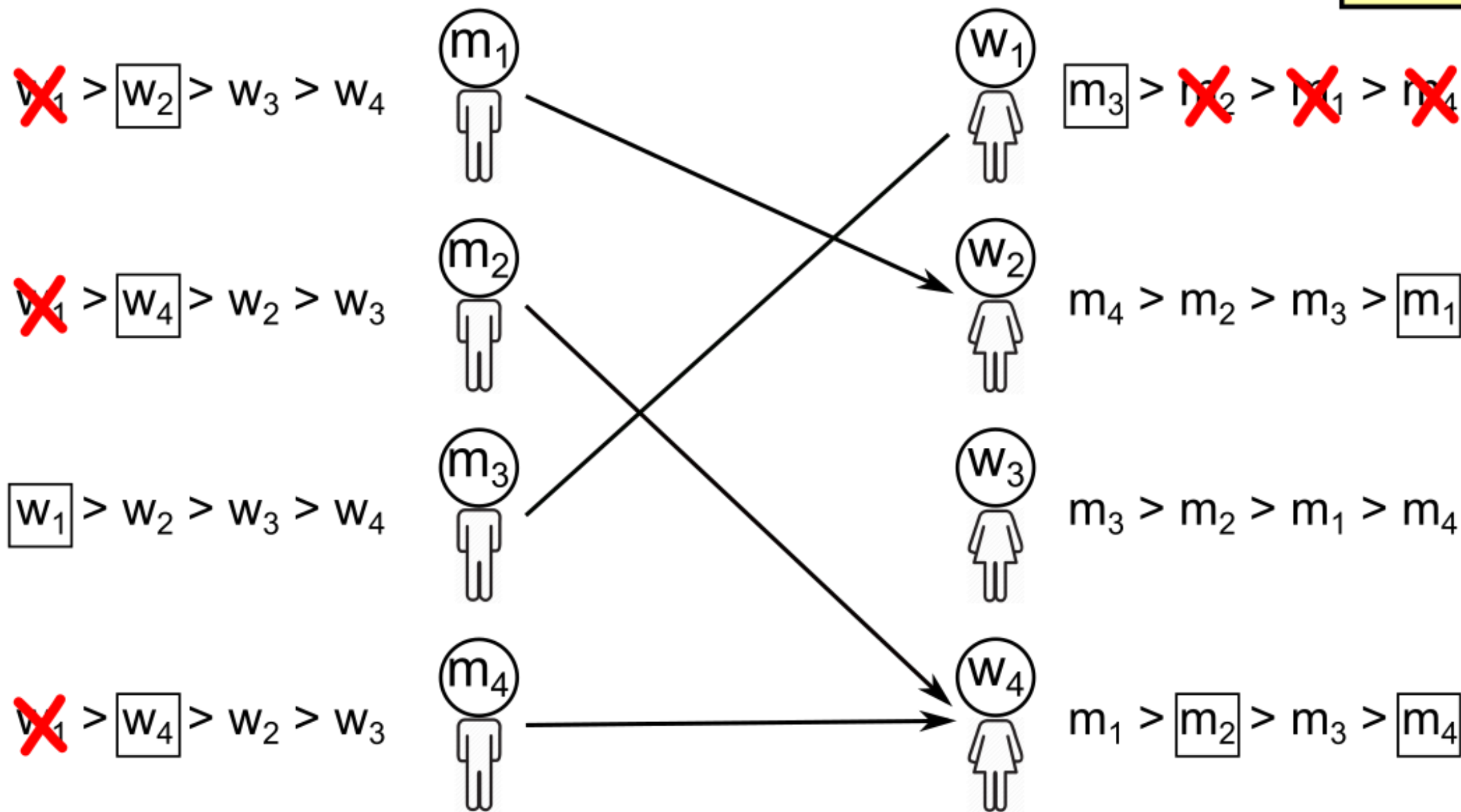
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Round 2



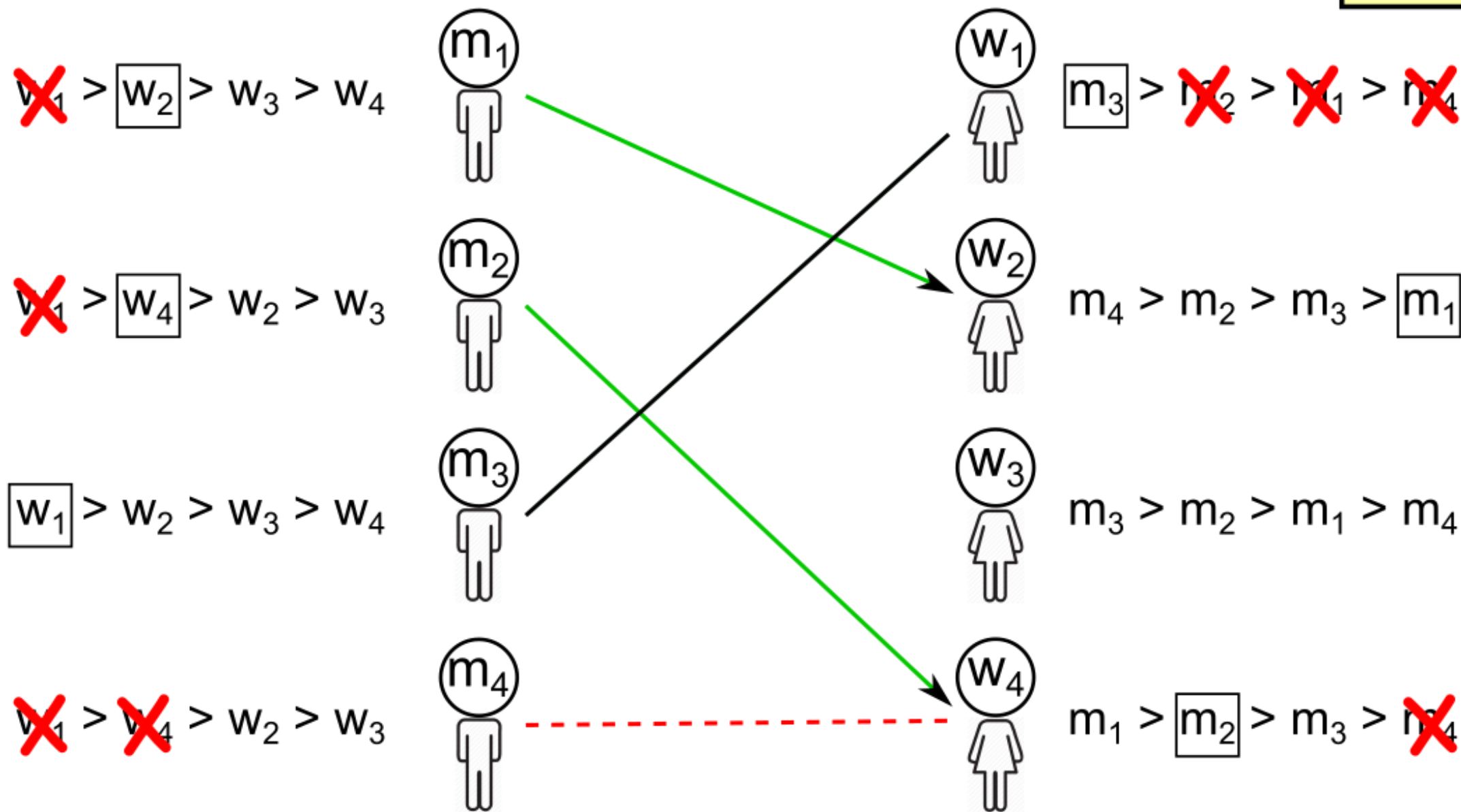
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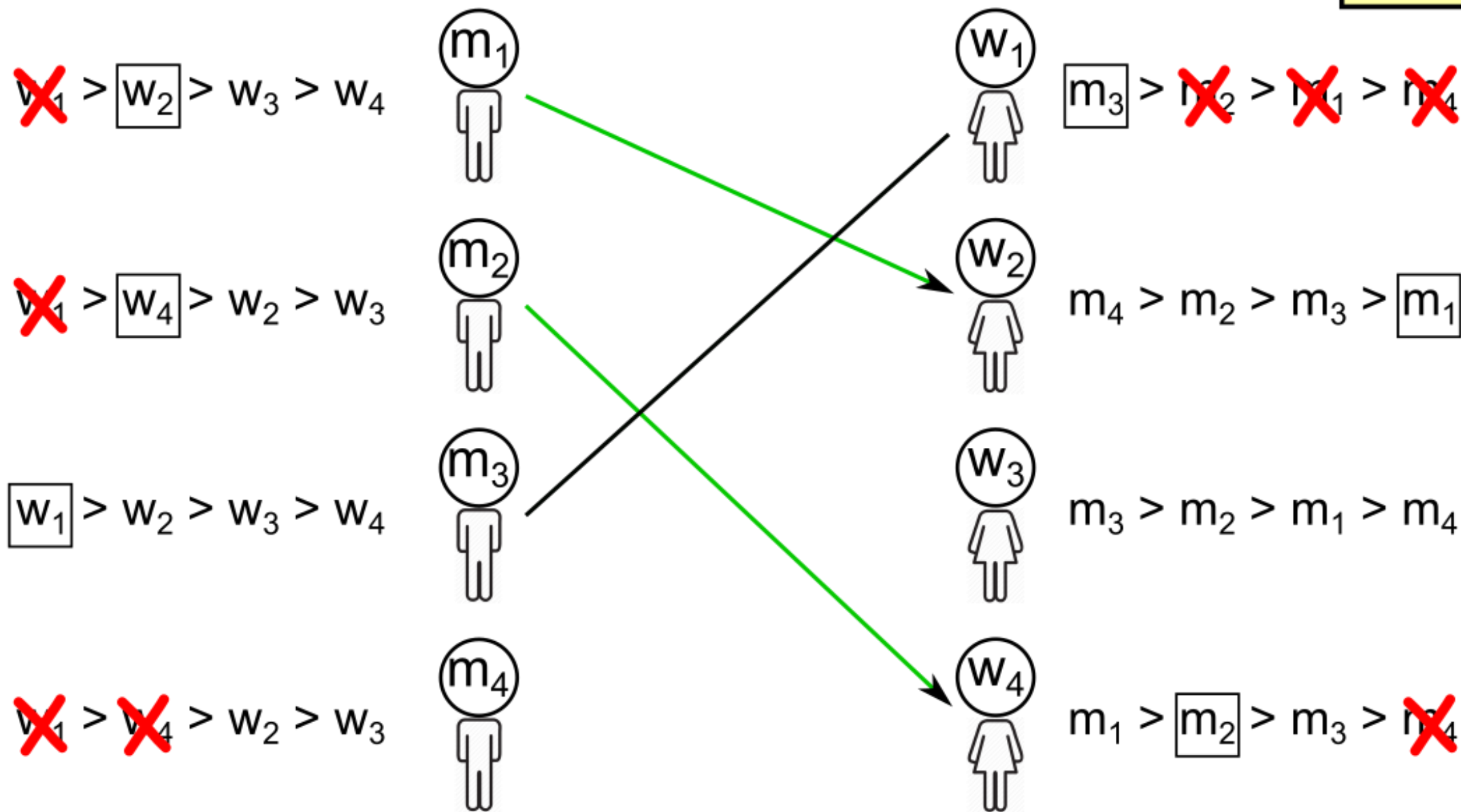
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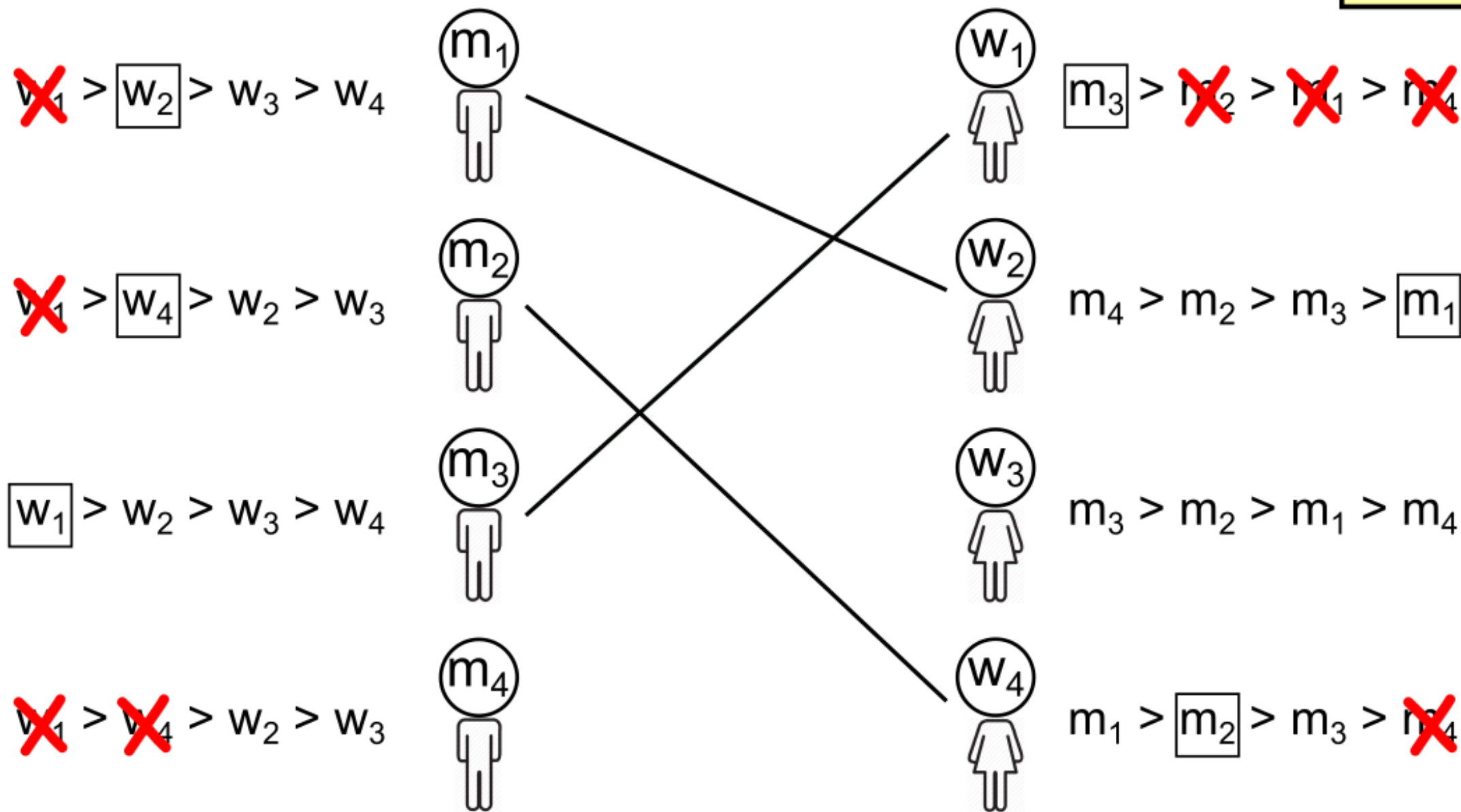
Round 2





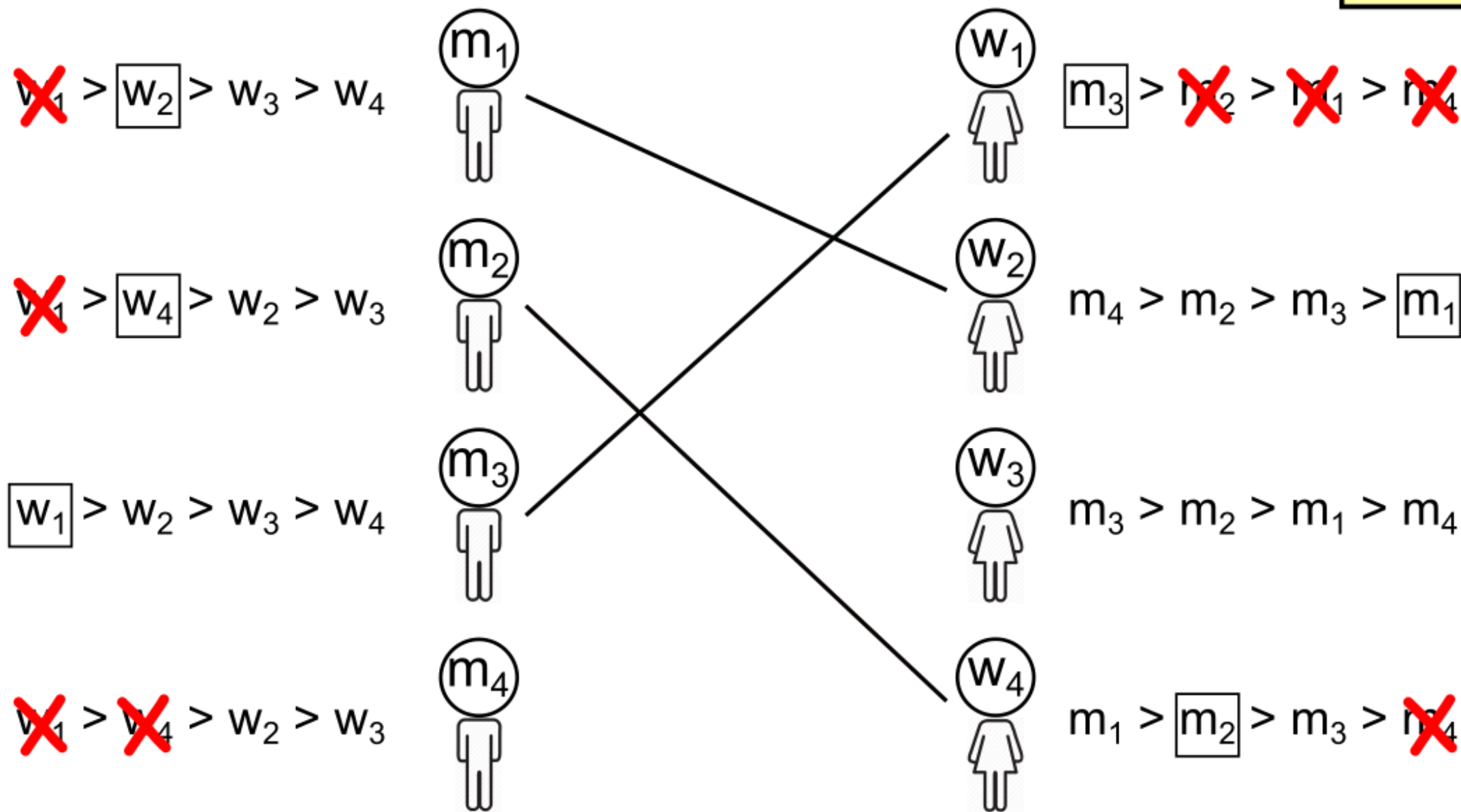
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Round 2



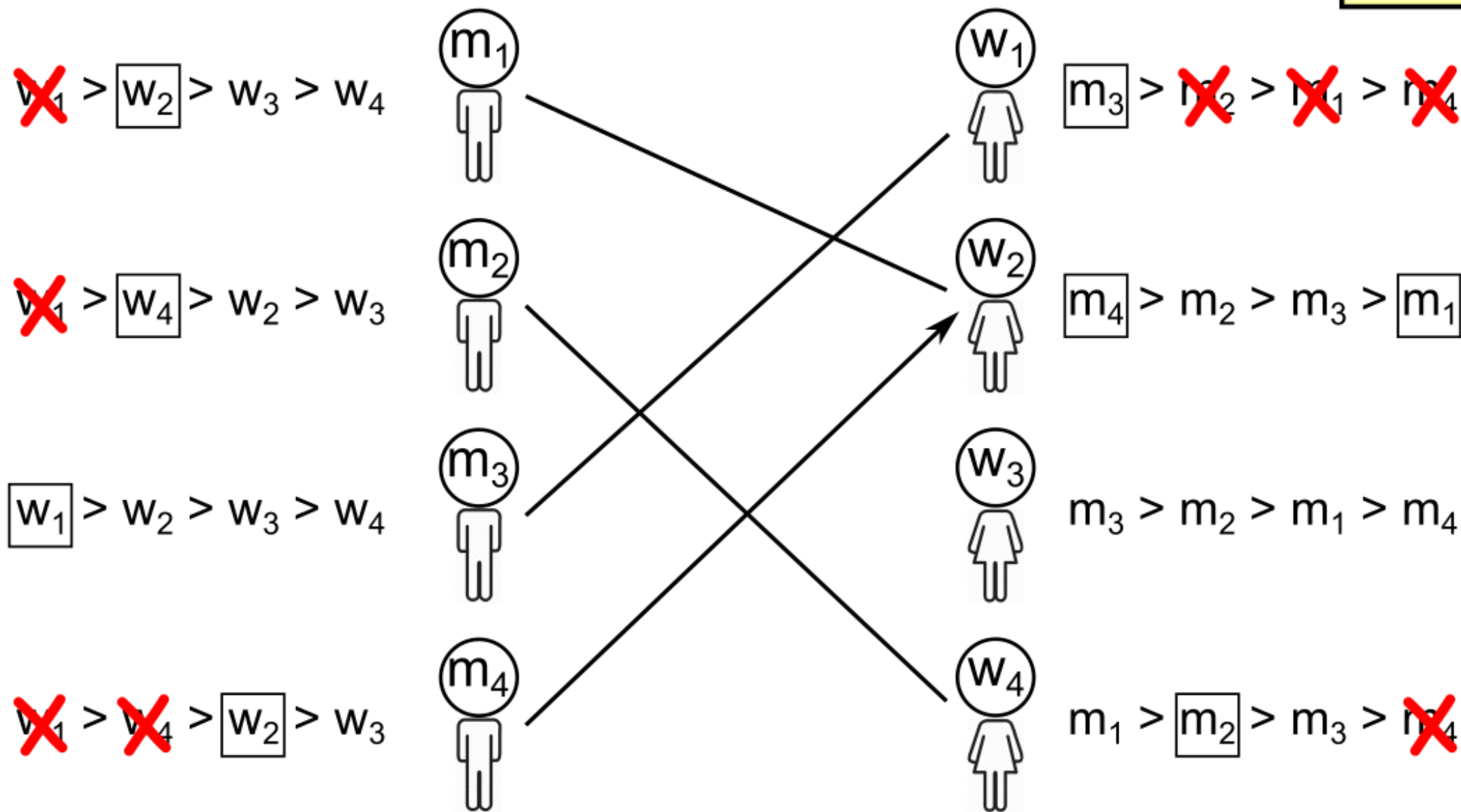
# Deferred-Acceptance Algorithm

Round 3



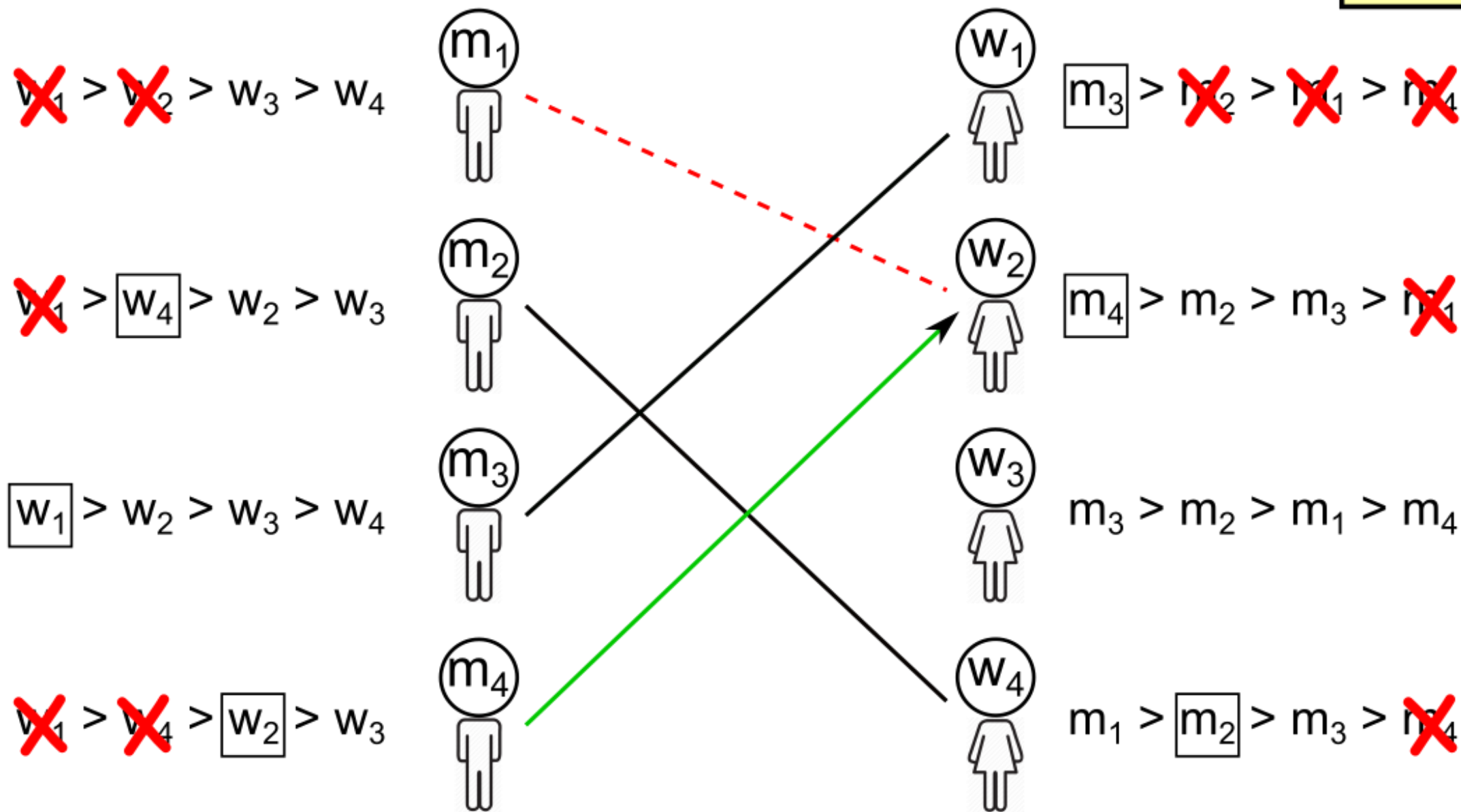
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Round 3



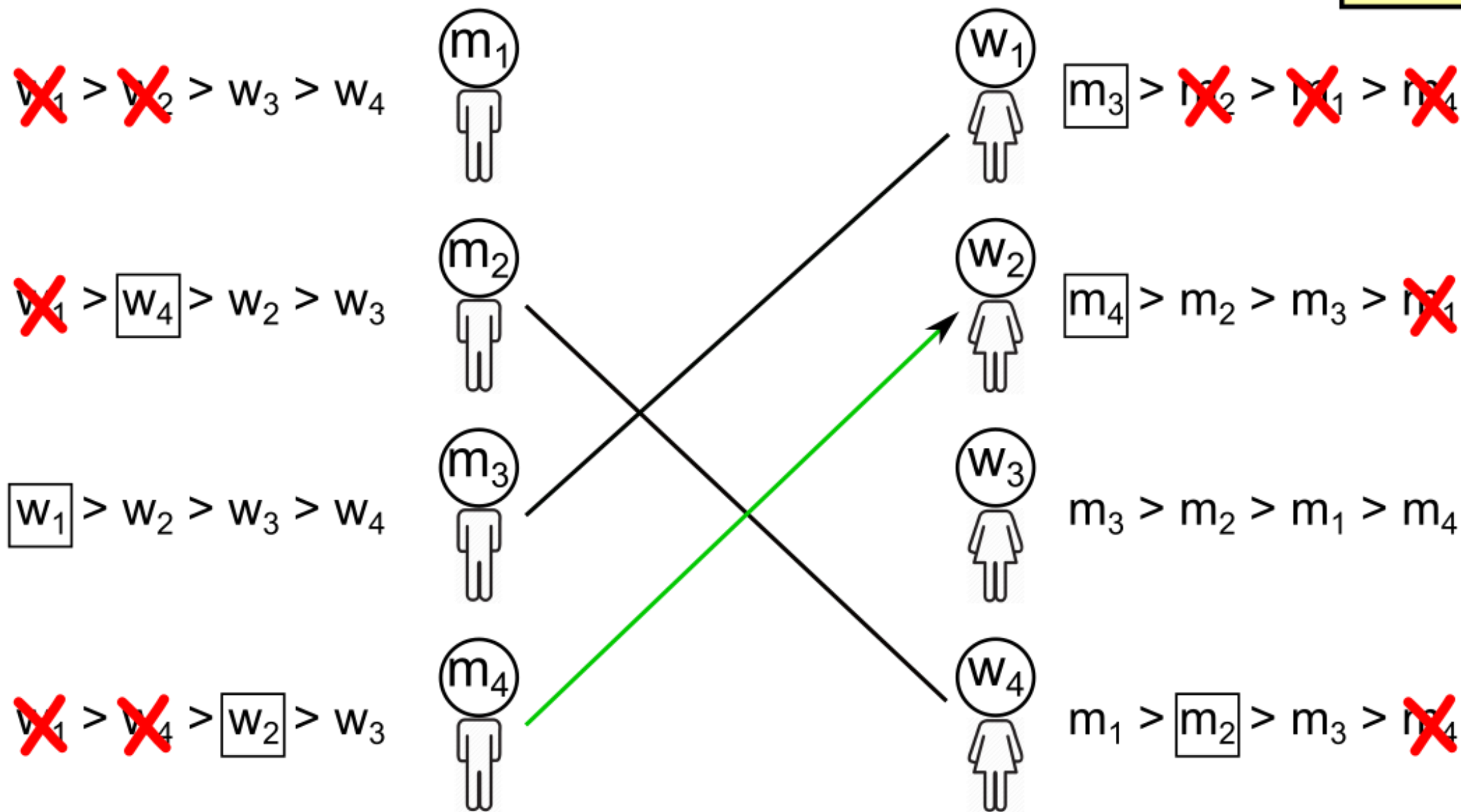
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Round 3



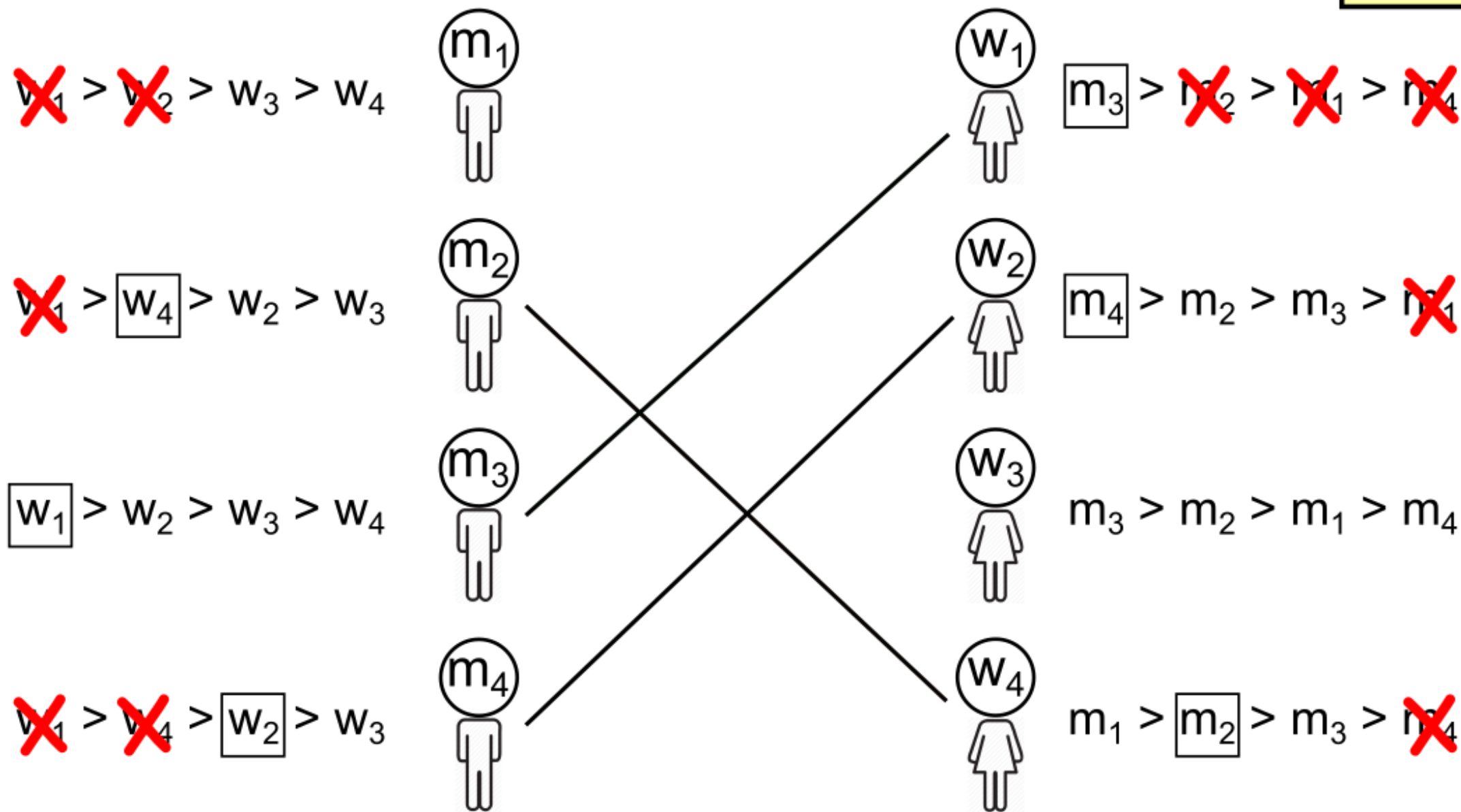
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Round 3



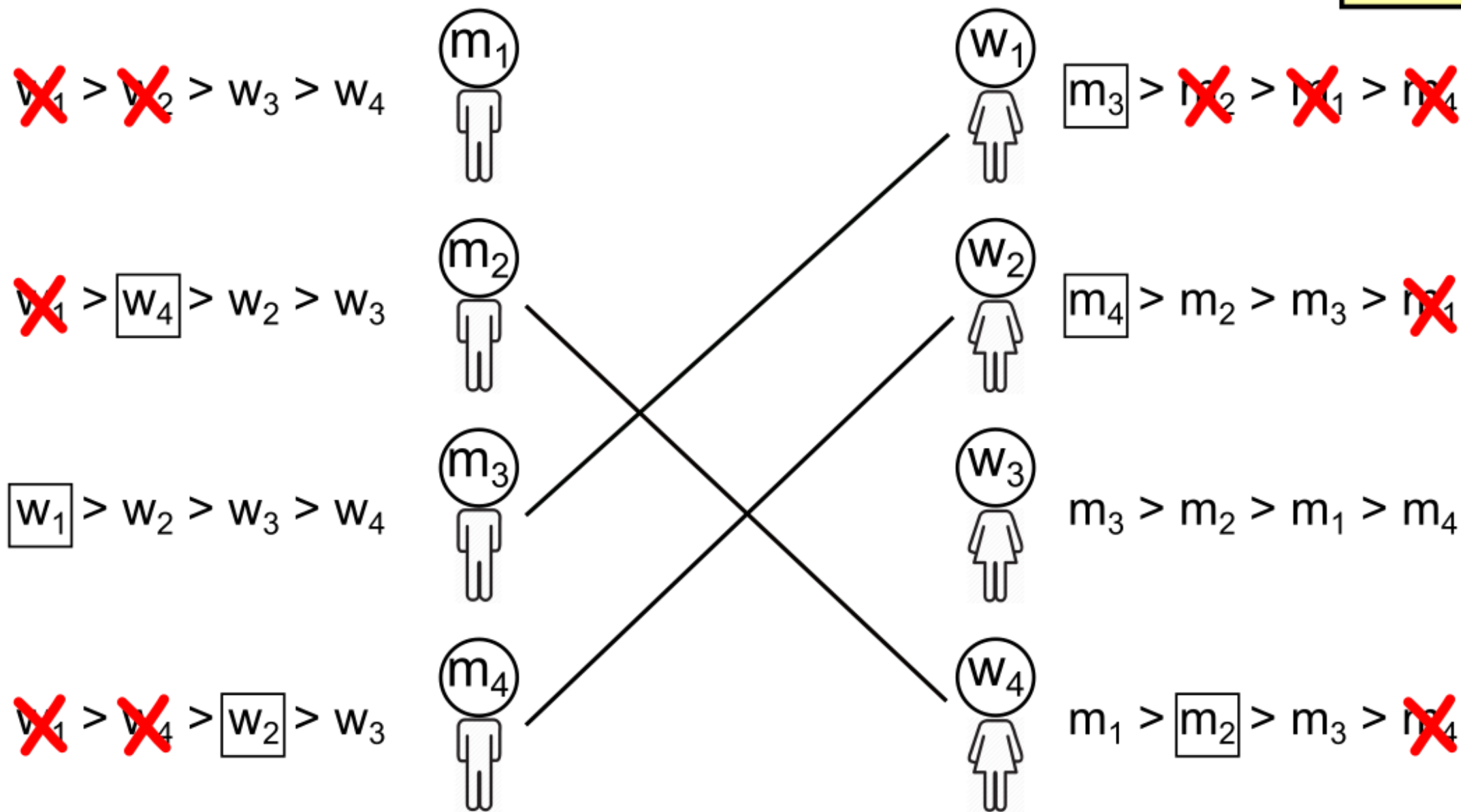
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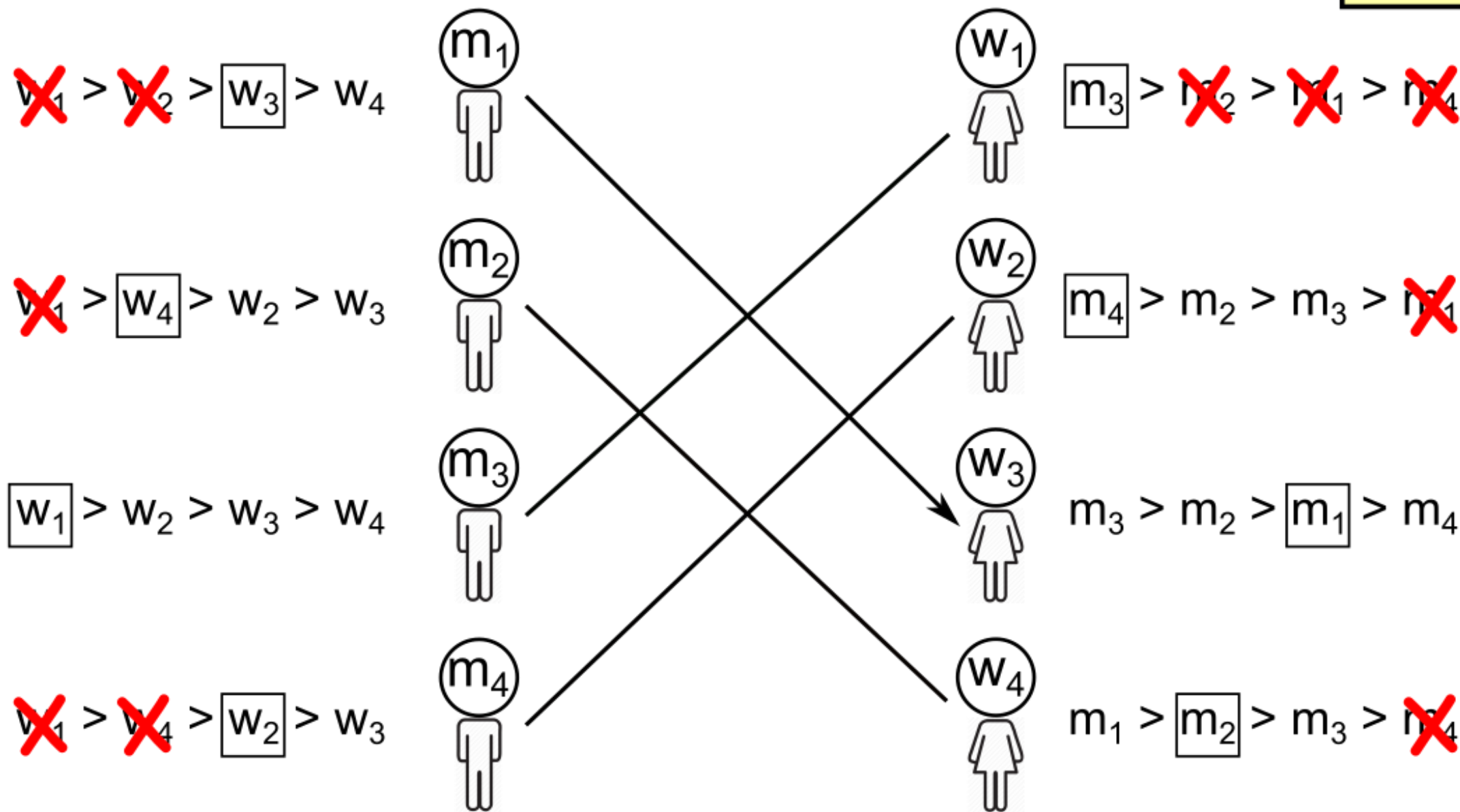
# Deferred-Acceptance Algorithm

Round 4



# Deferred-Acceptance Algorithm

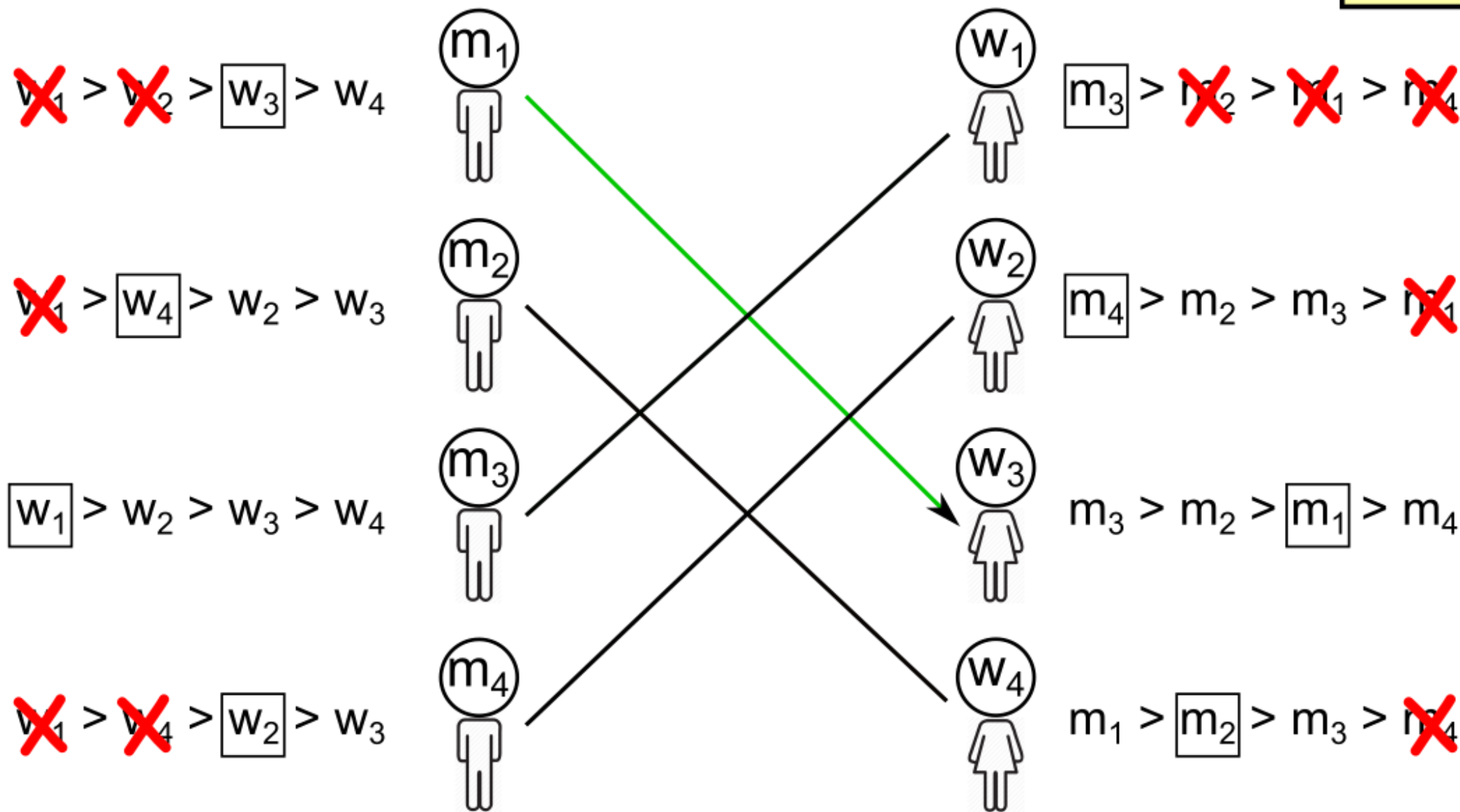
Round 4





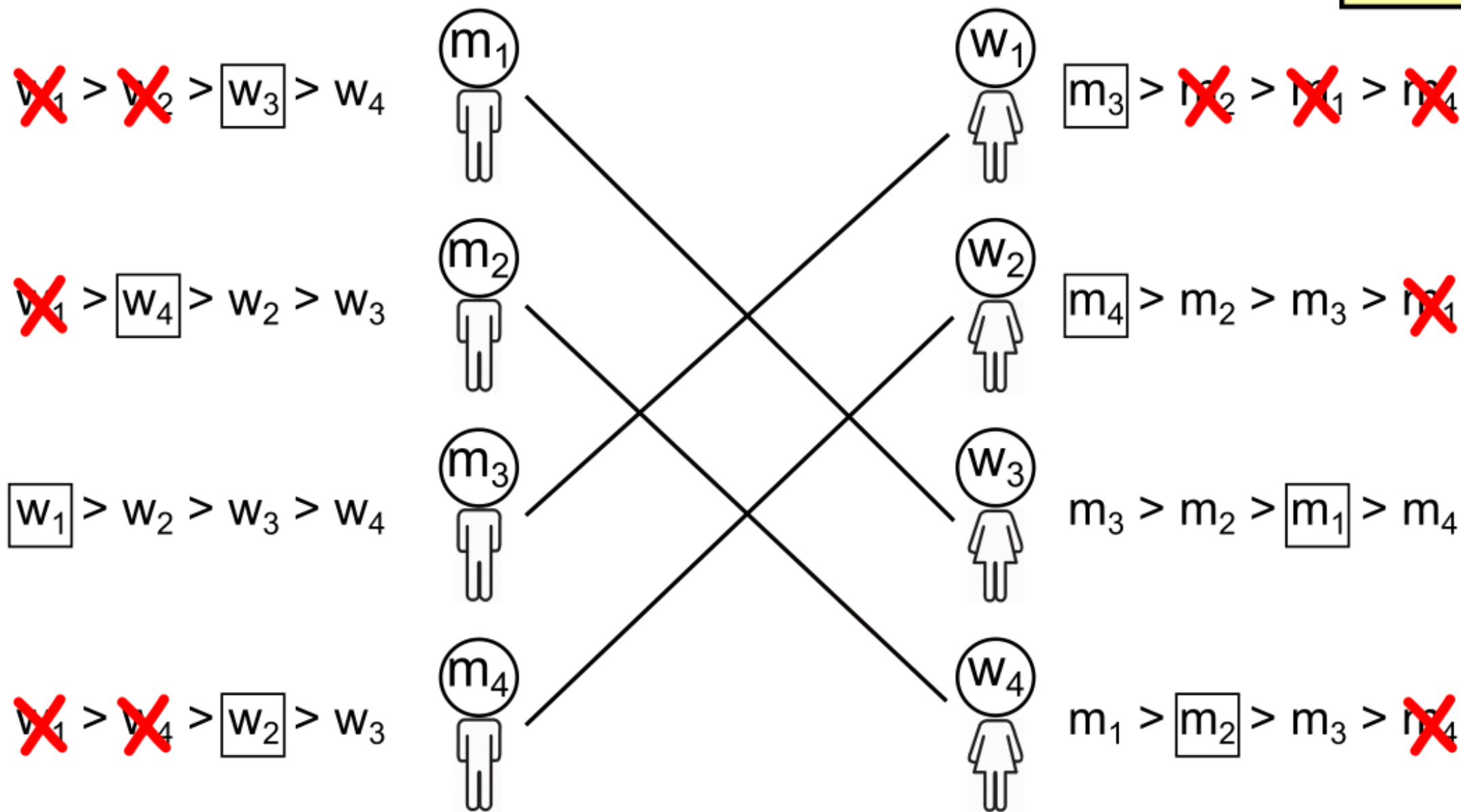
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Round 4

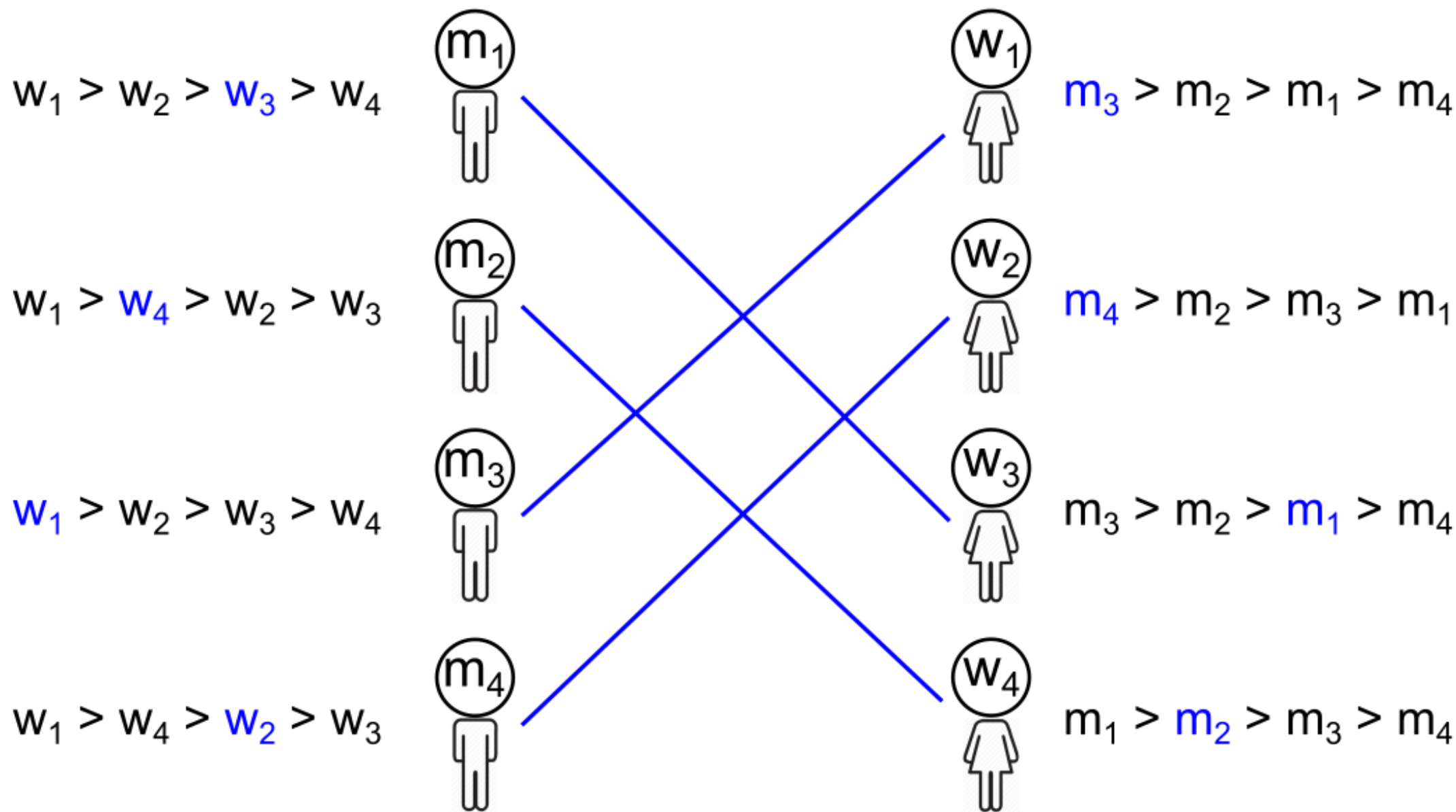


# Deferred-Acceptance Algorithm

Round 4



# Deferred-Acceptance Algorithm





1 Does the deferred-acceptance algorithm always terminate?

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In each round, at least one proposal is made.

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Deferred-acceptance algorithm terminates in **polynomial time**.

2

Does the deferred-acceptance algorithm always find a perfect matching?

2

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Then, there must be an unmatched man  $m$ .

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Man  $m$  must have proposed to (and been rejected by) woman  $w$ ,  
meaning  $w$  got a better-than- $m$  proposal in some round.

Once tentatively matched, a woman never becomes unmatched.



3

Does the deferred-acceptance algorithm always find a stable matching?

3

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3

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Suppose the DA matching has a blocking pair  $(m, w)$ .

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Men make proposals in decreasing order of their preference.  
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Women only "trade up" during the DA algorithm.

# Applications



**HOSPITAL**









## CHAPTER 18

# Applications of Matching Models under Preferences

Péter Biró

Trends in Computational  
Social Choice

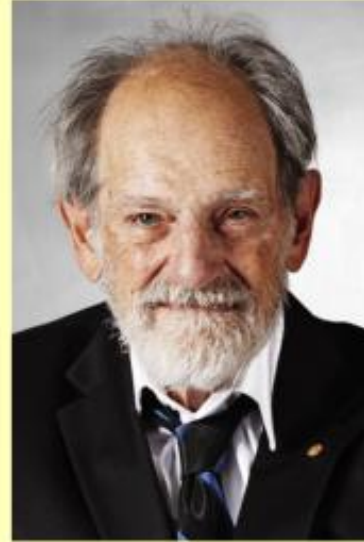
### 18.1 Introduction

Matching problems under preferences have been studied widely in mathematics, computer science and economics, starting with the seminal paper by Gale and Shapley (1962). A comprehensive survey on this topic was published also in Chapter 14 of the Handbook of Computational Social Choice (Klaus et al., 2016), and for the interested reader we recommend consulting the following four comprehensive books on the computational (Gusfield and Irving, 1989; Manlove, 2013) and game-theoretical, market design aspects (Roth and Sotomayor, 1990; Roth, 2015) of this topic. In this chapter our goal is to give a general overview of the related applications.

# Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012



Alvin E. Roth



Lloyd S. Shapley

*"for the theory of stable allocations  
and the practice of market design."*

# Logistics

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Assignments (4 x 10% = 40%)	Class participation (20%)	Project presentation and report (40%)
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Human readable; preferably in LaTeX

(5-6 problems, one *challenge*, ~one week)

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Ask questions! Get involved!  
Attendance  $\neq$  Participation

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Understand a paper (or a set of  
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+ Contribute *new* insights  
(Suggestive list on course website)

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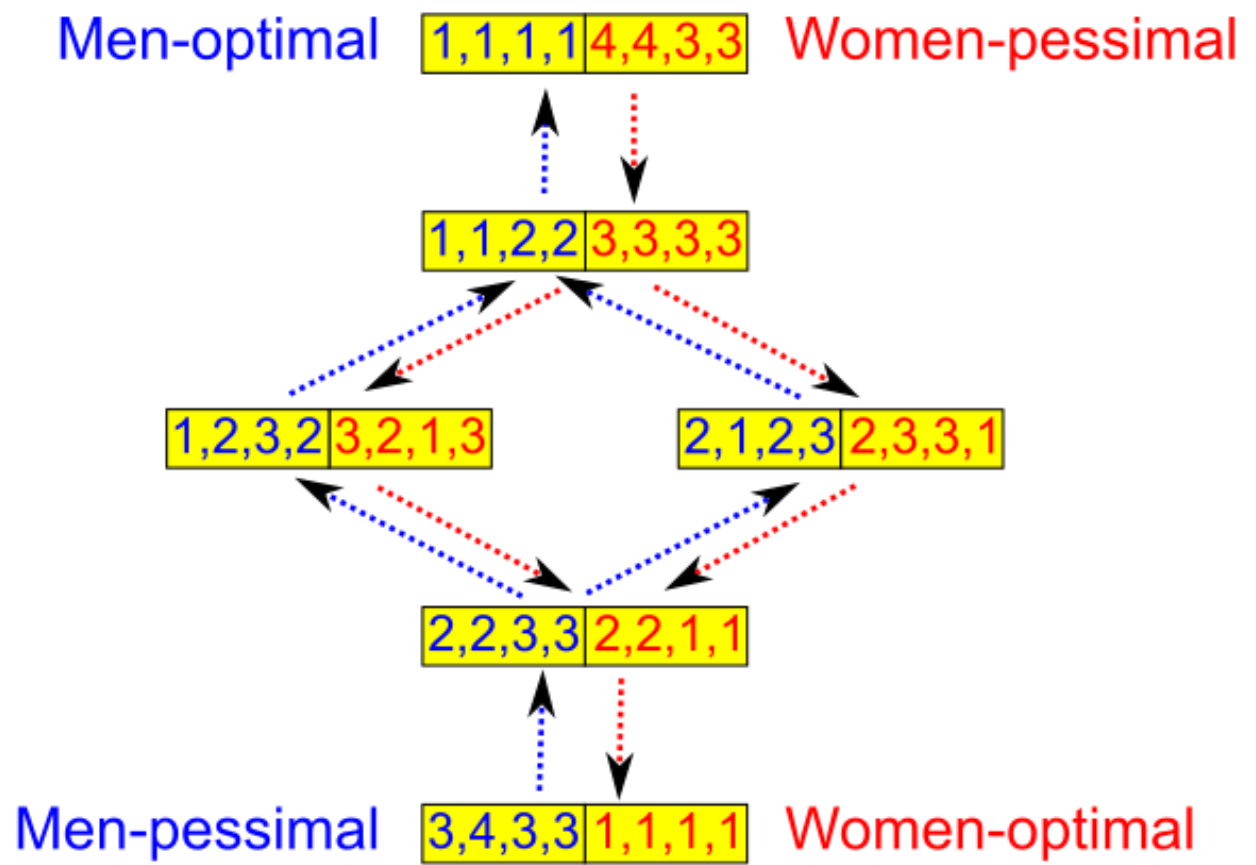
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# Next Time

## Structure of Stable Matchings



# References

- Stability and the Deferred Acceptance Algorithm

David Gale and Lloyd Shapley

“College Admissions and the Stability of Marriage”

*American Mathematical Monthly*, 69(1), 1962 pg 9-15



